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ACKNOWLEDGMENTS

This translation of S. M. Rytov's Theory of Electric Fluctuations and Thermal Radiation is being issued by the Air Force Cambridge Research Center because there is considerable current interest in the topics treated by Rytov and a comparable English text is lacking.

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CONTENTS

PREFACE

The theory of electrical fluctuations based on the application of statistics to macroscopic electrodynamics is presented. ~~in this monograph.~~ ~~Hence,~~ the numerous problems of the microtheory of electrical fluctuations in which the main role is played by parameters which characterize the behavior of microcharges are not discussed in this work. Nevertheless, the broad circle of questions — from the theory of thermal radiation in the optical region to the theory of electrical "noise" in quasi-stationary circuits — questions concerning the existence of matter not requiring insight into statistical electronics, is grasped from a single point of view. The general electrodynamic approach thus permits the study of problems in which ~~neither~~ the condition of being quasi-stationary ~~nor~~ or the condition of applicability of geometrical optics are not required (for instance, heat radiation in wave guides, cavity resonators).

Basic attention is given to the indicated formulation of the problem and to the development of the theory derived from it. This work therefore does not contain a systematic review of the existing literature on electrical fluctuations and does not pretend on a complete coverage of this literature.[1]

The monograph is directed to research workers and graduate students, interested in questions of electrical fluctuations and thermal radiation. With the exception of Chapter IV, the contents of this work, particularly Chapters III and V, can be of interest also to radio engineers dealing with thermal electrical "noise".

Derivations, necessary for a complete presentation but not of direct physical interest, are given in appendices. ~~at the end of the work.~~

It is my pleasant obligation to express my deep gratitude to Academician M. A. Leontovich for his constant attention and interest in relation to this work and for a number of detailed advices and instruc-

1. A detailed presentation of the theory and of experimental results in the field of electrical fluctuations and the literature up to 1936 is given in the monograph of V. L. Granovskii "Electrical Fluctuations" (M.-L., 1936).

tions given by him during many sessions in the course of realization of this work. I also want to thank Prof. G. S. Gorelik, who attracted my attention to research work related to the work of this book, and to Prof. V. L. Ginsburg and M. L. Levin, who made a number of valuable comments upon reading the manuscript.

INTRODUCTION

In the study of thermo-electrical fluctuations there are two classes of questions. The connection between these classes was clarified quite long ago. But up to the present time this connection was not formulated in the form of a unified theory.

One of the regions about which we are talking, and which is the considerably earlier one, deals with questions of thermal (or temperature) radiation. The attribution of thermal radiation to fluctuation phenomena has a well determined basis and reason. As was noted by M. A. Leontovich,[1] the division of physical phenomena into fluctuation and non-fluctuation phenomena acquires definite sense only upon indication of those magnitudes which characterize the phenomenon under study. If for such a magnitude we take the equilibrium energy of radiation U_ω, whose mean value $\bar{U}_\omega$ is given by Planck's equation, then the deviations $\Delta U_\omega = U_\omega - \bar{U}_\omega$ constitute the fluctuations, and the well-known expression for $\overline{(\Delta U_\omega)^2}$ obtained by Einstein[2] serves as a measure of the intensity of the fluctuations. If however emphasis is placed on the electrodynamic side of the phenomenon and the potentials E and H -- "linear" variations (linear relation between currents and charges) -- are used to describe the radiation field, then the mean density of equilibrium radiation energy itself

$$\bar{U}_\omega = \frac{1}{8\pi}\int_V (\overline{E^2} + \overline{H^2})\, dV$$

is a measure of the intensity of fluctuation, and particularly of the fluctuations E and H, whose mean value in this case equals zero. It is in this last sense that we attribute thermal radiation to electrical fluctuations.

In the theoretical treatment of questions arising here the methods of thermodynamics and statistical physics are applied to the electromagnetic

1. M. A. Leontovich, Introduction to Thermodynamics, M., 1951, Section 34. Also, G. S. Gorelik, UFN 44, 33, 1951.

2. A. Einstein, Phys. Zst. 10, 185, 817, 1909.

field, and the latter is studied by means of the approximations of geometrical optics. It is through this approximation that the laws of Kirchhoff for radiation are derived and the law of spectral distribution of equilibrium radiation has the same asymptotic nature established for radiation, which is included in a shell of sufficiently large sizes.

The other region is made up of thermo-electrical fluctuations in conductors, i.e., fluctuations of current strength, voltage, condenser charge, etc. The theory of these fluctuations, or, in the language of radio technology, of electrical "noises", either relies on the application of thermodynamics and statistics to quasi-stationary currents in circuits, consisting of linear (sufficiently thin) conductors, or is based on the statistical study of the behavior of microcharges in bodies.

The close connection between electrical "noises" and thermal radiation consists of the fact that this radiation constitutes an electromagnetic wave field caused by thermo-electrical fluctuations in bodies and in its turn acting upon the charges in these bodies.

The reason for the absence of a single and sufficiently general theoretical approach to such closely connected phenomena lies in the tremendously large difference in the frequencies of electromagnetic oscillations which constitute the interest in each of the indicated regions. The questions concerning thermal radiation originated and were studied as optical questions. Specifically the fact that oscillations of light frequencies were here the center of interest made it in many cases possible and sufficient to use the approximations of geometrical optics. On the other hand, electrical "noises" were experimentally discovered in the band of not too high (initially acoustical) radio frequencies. This made it possible to limit oneself in this investigation to the theory of quasi-stationary currents.

In recent times a considerable increase in the sensitivity of short-wave radio receiving apparatus was achieved, in large measure due to the development of radio location techniques permitting the capture of thermal electromagnetic radiation of bodies in the wave band from a meter to a centimeter. As is known, a new science -- radio-astronomy -- was born

from this technical basis. On the other hand, the study of electrical "noises" in conductors evolved in a natural way towards the side of higher frequencies. This was aided by questions concerning "noises" in non-stationary networks -- waveguides, cavity resonators, etc. -- and acquired actuality in connection with the perfection of radio-astronomy apparatus. Thus, the regions of thermal radiation and electric "noises" bordered directly on each other in the band of ultra-high radio frequencies. There is no doubt that, as the practically utilizable radio frequencies increase further, the unification of these regions will become more and more narrow.

The theory which would embrace from a single viewpoint thermal electric "noise" and thermal electro-magnetic radiation must be -- at least initially -- sufficiently general in two respects. On the one hand, it must not be restricted to any very detailed concept of the electrical microstructure of bodies, i.e., it must describe their electro-magnetic properties in as general a way as possible, for instance, by means of complex permittivities. On the other hand, the theory must not be bound neither by the condition of quasi-stationality nor by the approximation of geometrical optics, permitting thereby, too, the examination of questions in which the wavelengths are comparable to the body dimensions. We are thus talking about the applicability of statistical methods to general macroscopic electro-dynamics.

As is known, a necessary condition for the possibility of such an approach is the requirement that the non-homogeneities of the macro-fields (wavelengths) be much greater than the micro-non-homogeneities which are due to the molecular structure of bodies. This requirement is satisfied even for optical frequencies ω, i.e., the phenomenological approach can extend also to the region of quantum phenomena, for which $\hbar\omega \gg kT$.[1]

1. $\hbar$ is Planck's constant (1.05×10^{-27} erg sec.), k is Boltzmann's constant (1.38×10^{-16} erg/degree), T is the absolute temperature. For $T = 300^{\circ}$ the equality $\hbar\omega = kT$ occurs for a wavelength $\lambda \approx 0.05$ mm.

But this kind of a representation of the body matter, as well as of the continuum implies the renunciation of statistical electronics, i.e., of taking into account of such parameters as the elementary charge, the number of elementary charges per unit of volume, thermal velocities of microcharges, their free path, etc. Nevertheless, the indicated formulation of the question permits the examination and unification of a large circle of macroscopic problems, which are of theoretical and of practical interest.

The representation of the chaotic lateral field which is distributed over the volume of the bodies under consideration constitutes the basis for the application of statistics to general electrodynamics. The action of this field determines the fluctuations of the charge and of the current in bodies, and thereby also the corresponding fluctuation radiation.

In the region of quasi-stationary currents, the concept of the random electro-motive force $\mathcal{E}$ localized in the network, which causes the electrical fluctuations in this network, had been introduced by de Haas-Lorentz[1] by analogy to the random mechanical force which causes the Brownian motion of a particle[2]. Examining the circuit composed of the inductance L and the resistance R, de Haas-Lorentz obtained the formula

$$\overline{X^2} = 2RkT\tau$$

for the mean square "impulse" $X = \int_{\tau} \mathcal{E}\, dt$ of this emf for the time interval τ. Later, starting from the theorem of equi-partition of energy according to degrees of freedom, Schottky[3] evaluated the power of thermal "noise" in an oscillatory circuit as a function of its time of establishment, i.e., of its selectivity. However, the approach of operating successively on the spectral intensity of the random electromotive force

1. G. L. de Haas-Lorentz, Die Brownsche Bewegung und einige verwandte Erscheinungen. Die Wissenschaft 52, p 86 (Braunschweig, 1913).

2. P. Langevin, C. R. 146, 530, 1908.

3. W. Shottky, Ann. d. Phys. 57, 541, 1918.

acting in the active resistances of the network, was developed only in 1927[1] simultaneously with the experimental discovery of electric fluctuations in conductors.[2]

M. A. Leontovich and the author[3] have shown how to pass (whilst maintaining the conditions of quasi-stationality) from the integral electromotive force to the random lateral electric field determined by it and distributed in the material of the conductor and have given the form of the space correlation function for any one of the components of the spectral intensity of this field. The next step which is also made in this work consists of the generalization of the concept of the lateral field and its correlation function for the frequency not limited by the quasi-stationary condition. The lateral fluctuation field (in the general case -- both the electrical and magnetic) is introduced thereby on the basis of electrodynamic equations, as a result of which the absence of any limitations for the relation between wavelength and body dimensions is achieved.

Naturally, the question as to the advantages and possibilities of the theory relying on the indicated premises arises. It is, however, more expedient to put off such an examination of this question to the concluding section of this work. Temporarily, only a very summary and not especially clear exposition of it can be given.[4]

As had been said, the general electrodynamic basis of the theory permits the unification of the asymptotic laws of classical radiation theory, the laws of electric fluctuations in the quasi-stationary region and the intermediary problems of bodies, whose dimensions are comparable to wavelengths. From this same electrodynamic basis follows the regular methodical process which permits the formulation and solution of various fluctuation problems by general rules, as boundary problems of field

1. H. Nyquist, Phys. Rev. 29, 619, 1927; 32, 110, 1928.

2. J. B. Johnson, Nature 119, 50, 1927; Phys. Rev. 29, 367, 1927; 32, 97, 1928.

3. M. A. Leontovich and S. M. Rytov, ZETF 23, 246, 1952.

4. A brief summary of the results was published in DAN 87, 535, 1952.

theory. Understandably, if we speak of radiation, i.e., of waves giving energy transfer,[1] then a series of general energy results can be obtained by classical means. The latter reduces to an accounting of the asymptotic value of the number of degrees of freedom in a conservative system limited in some manner and to the application of the theorem of energy distribution according to degrees of freedom (in either its quantum or classical form). The classical method mentioned is utilized along with the electrodynamic derivations each time that this is possible and expedient. Its application is thereby considerably simplified because of the result known beforehand and because of the visual representation of the "mechanism" of the phenomenon under study, which is obtained with the help of the fluctuation electrodynamic theory.

Another legitimately begging question is this: how does one now solve problems of thermal radiation in the region of microwave radio engineering and does one not arrive, here, as a result of the absence of a general theory, at erroneous conclusions. Factually, what is done here is simply a transferral of Nyquist's reasoning concerning the principal wave in a two-conductor line to the H_{10} type wave usually used in waveguides. It is considered, in this case too, that the emitter, matched with the waveguide at the wave H_{10}, sends out into the waveguide in the frequency interval df the thermal power kTdf. This statement, of course, can easily be justified by means of simple energy considerations. In this work, it is obtained as a consequence of general initial premises, and one can therefore say that we are giving in this work only another one of its justifications.

However, the problems related to thermal radiation in the microwave region can be more varied and not so simple. Thus, for instance, one can talk of cases of multi-wave propagation in waveguides; of interference phenomena possible for a given band df in insufficiently long channels

1. We have in mind travelling waves, non-extinguishable in the absence of absorption, in contrast to supra-critical waves in waveguides and so-called non-uniform waves which arise in diffraction on too small structures, in propagation in media with imaginary index of refraction, and also in complete internal reflection.

containing unmatched elements; of questions related to the thermal electromagnetic field in the immediate vicinity of emitting surfaces (in particular -- in cavities small compared to wavelength) etc. A precise treatment of similar questions cannot be given within the limits of only energy relations and by taking into account only the propagating (subcritical) waves. In the theory under consideration all problems of this nature receive an exhaustive answer since their solutions are based on sufficiently general foundations.

THEORY OF ELECTRIC FLUCTUATIONS AND THERMAL RADIATION

CHAPTER I. GENERAL STATEMENTS

Section 1. Electrodynamic Premises

In the investigation of fluctuation phenomena we shall assume that all relations between various electric and magnetic magnitudes can be considered linear. This permits going over to the spectral expansion of all magnitudes according to time. At first we shall take the stand that the functions of time of interest to us satisfy the condition of expansion into a Fourier integral with respect to t

$$f(\vec{r},t) = \int_{-\infty}^{+\infty} g(\vec{r},\omega)e^{i\omega t}\, d\omega \tag{1.1}$$

and ' accordance with this we shall write electrodynamic equations for spectral amplitude densities $g(\vec{r},\omega)$.

Our aim is to describe, in a sufficiently general way without having recourse to the refinement of the microstructure of matter, the electric fluctuations occurring in matter. Particularly suitable for this purpose are the Lorentz equations for microfields, and we shall rely mainly on transferring the concepts of macroscopic electrodynamics to the region of micro-phenomena. Let us denote the electric and magnetic strengths of the micro-field by $\vec{E}$ and $\vec{H}$, and the densities of microchanges and microcurrents by ρ and $\vec{j}$. We have

$$\mathrm{curl}\,\vec{E} = -ik\vec{H} \tag{1.2}$$

$$\mathrm{curl}\,\vec{H} = ik\vec{E} + \frac{4\pi}{c}\vec{j}$$

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$$\operatorname{div} \vec{E} = 4\pi\rho$$

$$\operatorname{div} \vec{H} = 0$$

where $k = \frac{\omega}{c} = \frac{2\pi}{\lambda}$. To these equations, we must add relations between $\vec{j}$, $\vec{E}$ and $\vec{H}$ which, upon statistical averaging, become the so-called "equations of matter" of Maxwellian electrodynamics -- Ohm's Law and laws connecting polarization and magnetizeability (or electrical and magnetic induction) with macroscopic intensities.

We shall denote statistical averages (averaging over the whole) with a bar over the corresponding quantities. Then for the total macroscopic current we have

$$\overline{\vec{j}} = \frac{(\varepsilon - 1) i\omega}{4\pi} \overline{\vec{E}} + c \operatorname{curl} \frac{\mu - 1}{4\pi\mu} \overline{\vec{H}} \tag{1.3}$$

The first term expresses the generalization of Ohm's Law which relates the conduction current to the polarization current; the second term gives the mean density of molecular currents with c curl $\vec{I}$, in which the magnetizeability $\vec{I}$ is expressed in terms of the induction of the magnetic field $\vec{B} = \overline{\vec{H}}$. The permeabilities ε and μ, generally speaking, are complex, i.e., the medium can have electrical as well as magnetic losses

$$\varepsilon = \varepsilon' - i\varepsilon'' = \varepsilon(\vec{r}, \omega) \tag{1.4}$$

$$\mu = \mu' - i\mu'' = \mu(\vec{r}, \omega)$$

We now introduce what constitutes a basic step: the lateral electric and magnetic fields with strengths $\vec{K}$ and $\vec{M}$, added to $\vec{E}$ and $\vec{H}$ respectively which act on the microcharges and microcurrents. In the presence of external sources the mean values $\overline{\vec{K}}$ and $\overline{\vec{M}}$ determine the corresponding external currents, charges, and constants of polarization and magnetization. In the absence of external sources

$$\overline{\vec{K}} = 0 \quad ; \quad \overline{\vec{M}} = 0 \tag{1.5}$$

The expression for the microcurrent density which, by virtue of (1.5) passes upon averaging to (1.3), we take in the form

$$\vec{j} = \frac{(\varepsilon - 1)i\omega}{4\pi}(\vec{E} + \vec{K}) + c \operatorname{curl} \frac{\mu - 1}{4\pi\mu}(\vec{H} + \vec{M}) \tag{1.6}$$

The corresponding expression for the microcharge density which satisfies the continuity equation div $\vec{j} + i\omega\rho = 0$, will be

$$\rho = -\operatorname{div} \frac{\varepsilon - 1}{4\pi}(\vec{E} + \vec{K}) \tag{1.7}$$

Upon substituting (1.6) and (1.7) into equations (1.2), the latter become

$$\begin{aligned} \operatorname{curl} \vec{E} &= -ik\vec{H} \\ \operatorname{curl} \frac{\vec{H}}{\mu} &= ik\varepsilon\vec{E} + ik(\varepsilon - 1)\vec{K} + \operatorname{curl} \frac{\mu - 1}{\mu}\vec{M} \\ \operatorname{div} \varepsilon\vec{E} &= -\operatorname{div} (\varepsilon - 1)\vec{K} \\ \operatorname{div} \vec{H} &= 0 \end{aligned} \tag{1.8}$$

Understandably, the last two equations are a direct consequence of the first two and in future we shall not write them.

At first glance, expression (1.6) may appear somewhat strange: this is a relation for microquantities which nevertheless contains the macroscopic characteristics ε and μ of the medium. However, in the semiphenomenological theory developed here, this is completely understandable. In essence, (1.6) means that in the microcurrent $\vec{j}$ the terms $\frac{(\varepsilon - 1)i\omega}{4\pi}\vec{E} + c \operatorname{rot} \frac{\mu - 1}{4\pi\mu}\vec{H}$, which upon averaging give (1.3), are separated and everything else which may determine $\vec{j}$ is included in $\vec{K}$ and $\vec{M}$, whose mean values, in the absence of external sources, are equal to zero.

Another question which naturally arises concerns this: is it necessary to introduce two lateral fields, $\vec{K}$ and $\vec{M}$, since only a certain combination of them appears in the basic equations (1.8), i.e., the

summed fluctuation "strength" $ik(\epsilon - 1)\vec{K} + \text{curl}\,\frac{\mu - 1}{\mu}\vec{M}$. This question cannot be resolved without recourse to the statistical properties of $\vec{K}$ and $\vec{M}$. Using only equations (1.8), without any supplementary concepts, we can vary the value of the micropotential in such a way that the lateral fluctuation "strength" will be present either in one of these basic equations or in both. If, for example, instead of $\vec{H}$, for which $\overline{\vec{H}} = \vec{B}$, we introduce $\vec{H}' = \vec{H} - \frac{\mu - 1}{\mu}(\vec{H} + \vec{M}) = \frac{\vec{H}}{\mu} - \frac{\mu - 1}{\mu}\vec{M}$, for which $\overline{\vec{H}'} = \vec{B} - 4\pi\vec{I}$ (i.e., $\overline{\vec{H}'}$ is the macroscopic magnetic field-strength and not its induction), then the basic equations (1.8) take the following form

$$\begin{aligned} \text{curl}\,\vec{E} &= -ik\mu\vec{H}' - ik(\mu - 1)\vec{M} \\ \text{curl}\,\vec{H}' &= ik\epsilon\vec{E} + ik(\epsilon - 1)\vec{K} \end{aligned} \tag{1.9}$$

We shall convince ourselves later that as far as the form of the correlation function of lateral fields is concerned the symmetrical form (1.9) is the more meaningful one (Section 9).

Similarly we shall see that the introduction of each of the lateral fluctuation fields $\vec{K}$ and $\vec{M}$ is necessarily related to the complexity of the corresponding ϵ and μ, i.e., to the presence of electric and magnetic losses. If magnetic losses are absent (μ - real), then $\vec{M} = 0$ and the equations are

$$\begin{aligned} \text{curl}\,\vec{E} &= -ik\mu\vec{H} \\ \text{curl}\,\vec{H} &= ik\epsilon\vec{E} + ik(\epsilon - 1)\vec{K} \end{aligned} \tag{1.10}$$

By $\hat{H}$ we understand here, and in what is to follow, the vector $\vec{H}'$ which had been introduced in the writing of the symmetrical equations (1.9). We shall study equations (1.10) right up to Section 9.

It is necessary to note that the representation (1.6) for the "electrical" part of the total microcurrent (first term) does not follow the generally accepted method of introducing the lateral electric field into macroscopic electrodynamics. Usually this field is

not introduced into the generalization but into the simple Ohm's Law, i.e., only into the expression for the conduction current:

$$\vec{j}_{\text{cond.}} = \sigma(\vec{E} + \vec{K}_0) = \frac{\varepsilon''\omega}{4\pi}(\vec{E} + \vec{K}_0)$$

With this kind of an approach, one would have to write for the electric part of the microcurrent, instead of the first term of (1.6), the expression

$$\vec{j} = \frac{(\varepsilon'-1)i\omega}{4\pi}\vec{E} + \frac{\varepsilon''\omega}{4\pi}(\vec{E} + \vec{K}_0) = \frac{(\varepsilon-1)i\omega}{4\pi}\vec{E} + \frac{\varepsilon''\omega}{4\pi}\vec{K}_0 \qquad (1.11)$$

As is seen by comparing with (1.6), the lateral field strength $\vec{K}_0$ introduced in this way is related to our $\vec{K}$ as follows

$$\vec{K}_0 = \frac{i(\varepsilon-1)}{\varepsilon''}\vec{K}. \qquad (1.12)$$

Expression (1.11) is favored, not only because of the fact that this presentation corresponds to the usual way of introducing the lateral field but also because of the fact that this field, being immediately introduced only into the "noise" part of the current, which is related to the losses, has a rather simple correlation function (Section 6).

Nevertheless, the presentation (1.6) chosen by us has well known advantages. First of all, in it, no basic differentiation is made beforehand between the bound and free charges, i.e.. fluctuations of conduction as well as of polarization currents are taken into account; this is completely evident from the point of view of the microfield theory. Similarly the question whether only that part of the current which is related to the losses, makes "noise" is not decided beforehand -- this resolves itself later. Secondly, in (1.6) there is a certain symmetry in the representations of the "electric" and "magnetic" parts, i.e., in the writing of the first and second terms which express the density of molecular currents. The representation of the second term in a form analogous to (1.11) would require the introduction

of the lateral magnetic field $\vec{H}_0$ only in connection with magnetic losses (i.e., in the presence of μ''), which, to an even greater extent than in the case of the first term, depends on the results to follow.

In the solution of electrodynamic equations for piecewise continuous media, we have the usual boundary conditions which express the continuity of the tangential components of $\vec{E}$ and $\vec{H}$

$$[\vec{N}, \vec{E}_1 - \vec{E}_2] = 0 \quad ; \quad [\vec{N}, \vec{H}_1 - \vec{H}_2] = 0 \tag{1.13}$$

where $\vec{N}$ is the unit vector of the normal to the separating boundary, directed from medium 1 into medium 2. However, for media with a very large imaginary part in the complex index of refraction $\sqrt{\varepsilon\mu}$ (highly developed skin-effect), it is expedient to use relations based on the approximate boundary conditions of Leontovich. For this, the lateral fields, distributed in the general case over the entire space of the media under consideration, can equally well be replaced by certain surface lateral fields, entering not into the equations, but into the boundary conditions. Chapter III is devoted to the formulation and application of this problem.

Section 2. Reduced Spectral Intensities

It had been assumed above that the functions of time to be studied can be represented by Fourier integrals (1.1). In actual fact this is not so since we are interested in uniform fluctuation processes for which expansion (1.1) loses meaning and we can talk only about the so-called reduced spectral intensities[1] for the time-averaged quantities, bilinear with respect to the function f. It is precisely these reduced intensities, or, as they are called in the theory of electric "noises", densities of the power spectrum, which are of immediate physical

1. See for instance, A. M. Iaglom, 'Introduction to the Theory of Stationary Random Functions'. Usp. Mat. Nauk, 7, 5 (51) 1952; L. Landau and E. Lifshitz, Statistical Physics, Section 117 (GI TTL, 1951).

interest. Nevertheless, the writing of spectral expansions in the form (1.1) and the electrodynamic equations for spectral amplitudes resulting therefrom can formally be preserved, which is of great importance in view of the advantages which result from operating with these equations. The study of the problem given below does not in any way pretend to mathematical rigor and serves only for visual clarification.

Let T be the time interval many times larger than any correlation time which may be encountered in the uniformly random processes to be studied. We can then -- practically without any influence on the physical results -- consider the process to be periodical with period T and represent it in the form of a Fourier series

$$f(t) = \sum_{n=-\infty}^{+\infty} a_n e^{i\omega_n t} \quad ; \quad \omega_n = \frac{2\pi n}{T} \tag{2.1}$$

in which

$$a_n = \frac{1}{T} \int_{-T/2}^{+T/2} f(t)\, e^{-i\omega_n t}\, dt. \tag{2.2}$$

Without loss of generality we can consider the function $f(t)$ to be real so that a_n fulfills the relation

$$a_n = a^*_{-n} .$$

For the time average (i.e., T-average) of the product of two such functions $f(t)$ and $g(t)$ we have

$$\overline{f(t)g(t)}^t = \frac{1}{T} \int_{-T/2}^{+T/2} f(t)g(t)dt = \sum_{n,m} a_n b^*_m \delta_{nm} =$$

$$= \sum_{-\infty}^{+\infty} a_n b^*_n = \frac{1}{2} \sum_{-\infty}^{+\infty} (a_n b^*_n + a^*_n b_n) . \tag{2.3}$$

We shall now increase T without limit, at the same time fixing the greatest n, so that the frequency interval from $\omega = 2\pi n/T$ to $\omega + d\omega = 2\pi(n + dn)/T$ will be fixed. This procedure, often applied when passing from the series to the Fourier integral,[1] requires the assumption that the following limit exists

$$\lim_{T\to\infty} \frac{Ta_n}{2\pi} = a(\omega). \tag{2.4}$$

By virtue of (2.2) the integrability of the absolute value of f(t) within the limits $(-\infty, +\infty)$ constitutes a sufficient condition for this. If (2.4) takes place, then the sum (2.1) passes in the limit to the Fourier integral

$$f(t) = \int_{-\infty}^{+\infty} a(\omega)\, e^{i\omega t}\, d\omega \tag{2.5}$$

and for the time average (2.3) we obtain

$$\lim_{T\to\infty} \overline{f(t)g(t)}^t \approx \lim_{T\to\infty} \frac{\pi}{T}\int_{-\infty}^{+\infty} \left\{a(\omega)b^*(\omega) + a^*(\omega)b(\omega)\right\} d\omega = 0$$

Matters are different for functions that are not expandable into Fourier integrals (not decreasing fast enough as $|t| \to \infty$), but whose average value (2.3) has a finite limit. Let

$$\lim_{T\to\infty} \frac{T}{4\pi}\left(a_n b_n^* + a_n^* b_n\right) = \vec{G}(\omega). \tag{2.6}$$

Then the limit (by assumption, existent) of the average value (2.3) will be

$$\lim_{T\to\infty} \overline{f(t)g(t)}^t = \int_{-\infty}^{+\infty} \vec{G}(\omega)\, d\omega. \tag{2.7}$$

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1. See for instance, R. Courant and D. Hilbert 'Methods of Mathematical Physics', vol. I, Chap. II, Section 6 (M. - L., 1951)

The quantity $G(\omega)$ is called reduced spectral intensity. When the condition (2.6) which, roughly speaking, means that for large values of n $a_n \sim b_n \sim \frac{1}{\sqrt{T}}$ is fulfilled, the passage from series (2.1) to the integral loses meaning.

Let us now assume that we are operating with the series (2.1) at very large but still finite values of T. Then, considering (2.5) merely as a conditional notation of the series (2.1) we can deal with the quantities $a(\omega)$ as we would with the series coefficients a_n. Any relation of the form

$$a(\omega)b^*(\omega) = \tilde{a}(\omega)\tilde{b}^*(\omega)$$

must then be understood in the sense

$$a_n b_n^* = \tilde{a}_n \tilde{b}_n^*$$

which in the limit gives the equation of the corresponding reduced spectral intensities. It is precisely this kind of a treatment that we are going to keep in mind, applying the formal expansion into the Fourier integral and using the spectral amplitude densities $\vec{E}$, $\vec{H}$, etc., for which all the electrodynamic equations have been written in Section 1.

Expression (2.3) can be rewritten in the form

$$\overline{f(t)g(t)}^t = \frac{a_0 b_0^* + a_0^* b_0}{2} + \sum_{n=1}^{\infty} (a_n b_n^* + a_n^* b_n) \qquad (2.8)$$

and, in particular,

$$\overline{f^2(t)}^t = a_0 a_0^* + 2 \sum_{n=1}^{\infty} a_n a_n^* \qquad (2.9)$$

i.e., the spectral intensities in the expansion for positive frequencies (these intensities we shall denote by the index ω) are twice the intensities in the expansion according to frequencies from $-\infty$ to $+\infty$.

Therefore, if

$$\vec{E}(t) = \int_{-\infty}^{+\infty} \vec{E}\, e^{i\omega t}\, d\omega \quad ; \quad \vec{H}(t) = \int_{-\infty}^{+\infty} \vec{H}\, e^{i\omega t}\, d\omega$$

Then according to (2.8) we shall have for the time-averaged vector of Umov-Poynting

$$S_\omega = \frac{c}{4\pi} \left\{ [\vec{E}, \vec{H}^*] + [\vec{E}^*, \vec{H}] \right\} \tag{2.10}$$

but for the time-averaged space densities of electric and magnetic energy the spectral intensities (in the absence of scattering) will be

$$\begin{aligned} u_{e\omega} &= \frac{\varepsilon + \varepsilon^*}{8\pi} \vec{E}\, \vec{E}^* \\ u_{m\omega} &= \frac{\mu + \mu^*}{8\pi} \vec{H}\, \vec{H}^* \end{aligned} \tag{2.11}$$

Section 3. Space Correlation of the Lateral Fluctuating Field

The solution of any electrodynamic problem attainable by means of equations (1.9) or (1.10) and of boundary conditions (1.13) leads to the representation of $\vec{E}$ and $\vec{H}$ in terms of $\vec{K}$ in the form of some linear operators of $\vec{K}$ (of volume integrals). The mean values of $\vec{E}$ and $\vec{H}$ are therefore evaluated from the mean values of the components of $\vec{K}$ and, in particular, in the absence of regular external sources are equal to zero. The fluctuations, however, of $\vec{E}$ and $\vec{H}$, and therefore the fluctuations of all magnitudes dependent on $\vec{E}$ and $\vec{H}$ are evaluated in terms of the random part of $\vec{K}$. We shall be interested in the mean energy values, i.e., values which are bilinear with respect to the components of $\vec{E}$ and $\vec{H}$. It is not difficult to imagine what particular characteristic of statistical properties of the random lateral field $\vec{K}$ is required for the evaluation of such quantities.

In the cross-multiplication of the components of $\vec{E}$ and $\vec{H}$, i.e., in the cross-multiplication of the corresponding linear operators of the components of $\vec{K}$, we shall obviously obtain under the sign of the resulting

operator (of the double volume integral) the product of these or those components taken at various points of the space. In this manner, in order to evaluate the statistical means from the bilinear quantities of interest to us, one must know the mean value of the product of the lateral field $\vec{K}$ at various points, i.e., the space correlation function of these components

$$F_{\alpha\beta}(\vec{r}',\vec{r}'') = \overline{\vec{K}_{\alpha}(\vec{r}'),\ \vec{K}''_{\beta}(\vec{r}'')} \quad ; \quad (\alpha,\beta = 1,2,3) \tag{3.1}$$

Initially, we shall examine the case of a homogeneous and isotropic medium, i.e., we shall consider ε and μ as scalar quantities, independent of the coordinates. One can then apply the same reasoning to $F_{\alpha\beta}$ as is used in hydrodynamics concerning the correlation function of relative velocities of liquid particles in homogeneous and isotropic turbulence.[1] It is the quantities $F_{\alpha\beta}$ which have to form a tensor and, because of the complete equality of the points of the medium, they must depend only on

$$\vec{r} = |\vec{r}' - \vec{r}''| .$$

The most general expression of the tensor dependent only on r, is

$$F_{\alpha\beta}(r) = f(r)\,\delta_{\alpha\beta} + g(r)\,\frac{x_{\alpha}x_{\beta}}{r^2} \tag{3.2}$$

where $\delta_{\alpha\beta}$ denotes the components of the unit tensor. Equation (3.2) can be written in a different form as follows:

$$F_{\alpha\beta}(r) = \Phi(r)\,\delta_{\alpha\beta} + \frac{\partial^2 \Psi}{\partial x_{\alpha}\partial x_{\beta}} \tag{3.3}$$

where

1. L. Landau and E. Lifschitz, Mechanics of Continuous Media, Section 27 (M. - L., 1944).

$$f(r) = \Phi + \frac{1}{r}\frac{d\Psi}{dr} \quad ; \quad g(r) = \frac{d^2\Psi}{dr^2} - \frac{1}{r}\frac{d\Psi}{dr}$$

If the homogeneous medium occupies a limited volume, the assumption that $F_{\alpha\beta}$ depends on $\vec{r}'$ and $\vec{r}''$ only through $r = |\vec{r}' - \vec{r}''|$ loses validity. This assumption can be preserved only for those points of the medium which are sufficiently far removed from the surface of the volume, i.e., removed by distances large compared to the correlation radius of the field $\vec{K}$. The following reasoning concerning this radius can be adduced.

For isotropic radiation of frequency ω, caused by external excitation in a given medium, the correlation radius would be of the order corresponding to the wavelength. But in $\vec{K}$ are included forces from any source, and, among them, forces related to the thermal movement of microcharges. One can therefore assume that the correlation radius of $\vec{K}$ has a magnitude of the order of the dimensions of the macro-inhomogeneities of the medium. The macroscopic theory to be studied assumes a priori that the micro-inhomogeneities of the medium are very small by comparison to the inhomogeneities of the macroscopic fields. Under such conditions, when even the wavelength in bodies and -- as we assume -- the dimensions of the bodies themselves are very large compared to the correlation radius of $\vec{K}$, it is natural to admit that this radius is simply equal to zero.

The second assumption consists of dropping the potential term in (3.3), i.e., we assume $\Psi(r) = 0$. Further below, we will be able to adduce some reasoning in favor of this assumption (Sections 8 and 26); for the moment, this assumption must be looked upon as a hypothesis which is justified only by the results obtained.

And so we make the following basic assumption concerning the form of $F_{\alpha\beta}$

$$\Phi(r) = C\delta(r) \equiv C\,\delta(x_1)\delta(x_2)\delta(x_3)$$

$$\Psi(r) = 0 \tag{3.4}$$

so that

$$r_{\alpha\beta}(r) = C\mathcal{E}_{\alpha\beta}\,\delta(\vec{r}), \tag{3.5}$$

In this manner, the components of $\vec{K}$ in the one and same direction possess the δ-correlation, but the components orthogonal to one another are not at all correlated. Since the correlation radius is equal to zero, expression (3.5) in the case of bounded bodies is valid throughout the space of the body, right up to the bounding surface. Concerning the constant C: it must be determined on the basis of supplementary physical requirements. This will be done in the next chapter.

In the presence of regular external sources, $\overline{\vec{E}}$ and $\overline{\vec{H}}$ are different from zero and represent macroscopic potentials [remember that by $\vec{H}$ the vector $\vec{H}'$ is understood, introduced when passing to equation (1.9)]. Hence,

$$\frac{\varepsilon + \varepsilon^{*}}{8\pi}\,\overline{\vec{E}}\cdot\overline{\vec{E}}^{*}$$

is the macroscopic dens. of electric energy and the quantity

$$\frac{\varepsilon + \varepsilon^{*}}{8\pi}\left\{\overline{\vec{E}\vec{E}^{*}} - \overline{\vec{E}}\cdot\overline{\vec{E}}^{*}\right\}$$

should serve as a measure for the intensity of fluctuation of the electric potential. But the first term of this expression is in fact the mean value of the true (microscopic) electrical energy density. According to the meaning itself of the lateral field $\vec{K}$ this is the total energy, into which enters not only the electric energy of the macrofield but also a part of the internal energy of the medium under study. This latter part is a finite quantity only insofar as the quantities characterizing the microstructure of the medium are different from zero. In the phenomenological method of describing the medium used by us, the only quantity of this type is the correlation radius of the lateral field $\vec{K}$.

We can therefore foresee that the admission of the δ-correlation for the components of $\vec{K}$ will have as a consequence the divergence of all expressions for the energy (mean bilinear) quantities calculable in the

interior of the absorbing medium. It is further clear that, although the introduction of a finite correlation radius, for instance the use of the correlation function $F_{\alpha\beta}$ in the form

$$F_{\alpha\beta}(r) = C\delta_{\alpha\beta}\frac{e^{-r^2/a^2}}{(\pi a^2)^{3/2}}, \tag{3.6}$$

assures the finiteness of the energy quantities in the medium, it nevertheless simply represents a formal procedure and cannot replace statistical electronics, which takes into account the actual peculiarities of the microstructure. Nonatheless, the analysis of results, obtained with the help of (3.6), is of certain interest for the clarification of concepts embodied in the basis of the theory (section 8).

Section 4. Correlation Function for Fourier Conjugates

In solving concrete problems it often appears to be convenient to use the expansion of $\vec{E}$, $\vec{H}$ and $\vec{K}$ according to some orthogonal function. It is then necessary to know the correlation function for the corresponding expansion coefficients of $\vec{K}$. In this connection we shall study two such cases, namely the expansion of $\vec{K}$ into the Fourier space integral for unbounded homogeneous media and for a semi-space $x_3 < 0$ filled with such media.

Let $\vec{K}$ be represented in the form of a triple Fourier integral.[1]

1. The space integrals to be studied in this section for unbounded or semi-infinite media, i.e., for functions which, strictly speaking, are not expandable into Fourier integrals, must be understood in the same restricted sense which was indicated in section 2 for spectral expansion with respect to time. One can in the same manner imagine that a multiplier is introduced into $\vec{K}$ giving a sufficiently rapid decrease of the integrand as $r \to \infty$ and that in the final results the corresponding "exponent of extinction" approaches zero.

where, for conciseness, the following has been introduced

$$\delta(\vec{p}) = \delta(p_1)\delta(p_2)\delta(p_3) \quad ; \quad \vec{p} = \vec{p}' - \vec{p}''$$

In this manner, the correlation function of components $\vec{g}$ has the form $A_{\alpha\beta}(\vec{p}')\delta(\vec{p}' - \vec{p}'')$, i.e., it constitutes a δ-correlation independent of any further assumptions concerning $F_{\alpha\beta}(r)$, which results in a greater advantage for evaluations.

It is not difficult to establish the form of the multiplier on $\delta(p)$ in (4.7), starting from the general representation (3.2) of $F_{\alpha\beta}$. We have

$$\int_{-\infty}^{+\infty} F_{\alpha\beta}(r)\, e^{-i\vec{p}'\cdot\vec{r}}\, d\vec{r} = \int_{-\infty}^{+\infty}\left\{f(r)\,\delta_{\alpha\beta} + g(r)\,\frac{r_\alpha x_\beta}{r^2}\right\} e^{-i\vec{p}'\cdot\vec{r}}\, d\vec{r} =$$

$$= \delta_{\alpha\beta}\int_{-\infty}^{+\infty} f(r)\, e^{-i\vec{p}'\cdot\vec{r}}\, d\vec{r} - \frac{\partial^2}{\partial p'_\alpha\, \partial p'_\beta}\int_{-\infty}^{+\infty}\frac{g(r)}{r^2}\, e^{-i\vec{p}'\cdot\vec{r}}\, d\vec{r}$$

Entering in here are integrals which are seen to be only functions of modulus $\vec{p}'$. Making use of their relationship through $u(p')$ and $v(p')$ we get

$$\int_{-\infty}^{+\infty} F_{\alpha\beta}(r)\, e^{-i\vec{p}'\cdot\vec{r}}\, d\vec{r} = u(\vec{p}')\,\delta_{\alpha\beta} - \frac{\partial^2 v(\vec{p}')}{\partial p'_\alpha\, \partial p'_\beta} =$$

$$= \left(u - \frac{1}{p'}\frac{dv}{dp'}\right)\delta_{\alpha\beta} - \left(\frac{d^2v}{dp'^2} - \frac{1}{p'}\frac{dv}{dp'}\right)\frac{p'_\alpha p'_\beta}{p'^2}$$

Finally

$$G_{\alpha\beta}(\vec{p}', \vec{p}'') = \frac{\delta(p)}{(2\pi)^3}\left\{U(p')\,\delta_{\alpha\beta} - V(p')\,\frac{p'_\alpha p'_\beta}{p'^2}\right\} \qquad (4.8)$$

where

$$U(p') = u - \frac{1}{p'}\frac{dv}{dp'} \quad ; \quad V(p') = \frac{d^2v}{dp'^2} - \frac{1}{p'}\frac{dv}{dp'} .$$

Analogous evaluations, carried out by substituting (4.8) into (4.5),

return us to expression (3.3) for $F_{\alpha\beta}$

$$F_{\alpha\beta}(r) = \Phi(r)\,\delta_{\alpha\beta} + \frac{\partial^2\Psi(r)}{\partial x_\alpha \partial x_\beta}$$

where

$$\Phi(r) = \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} U(p')\, e^{i\vec{p}'\cdot\vec{r}}\, d\vec{p}'$$

$$\Psi(r) = \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} \frac{V(p')}{p'^2}\, e^{i\vec{p}'\cdot\vec{r}}\, d\vec{p}' \qquad (4.9)$$

From this it follows that assumptions (3.4) made above concerning the form of the correlation function of the components of $\vec{K}$, namely

$$\Phi = C\delta(r) \quad , \quad \Psi = 0$$

mean that, in the correlation functions for $\vec{g}$, Fourier-conjugates of $\vec{K}$, we have

$$U = C \quad , \quad V = 0$$

i.e., $(\vec{p} = \vec{p}' - \vec{p}'')$

$$G_{\alpha\beta}(\vec{p}', \vec{p}'') = G_{\alpha\beta}(p) = \frac{C\delta_{\alpha\beta}}{(2\pi)^3}\delta(p). \qquad (4.10)$$

Taking into account the finiteness of the correlation radius with the help of (3.6), where

$$\Phi = C\,\frac{e^{-r^2/a^2}}{(\pi a^2)^{3/2}} \quad , \quad \Psi = 0$$

we obtain, as can be readily seen with the help of (4.7), the expression for $G_{\alpha\beta}$

$$G_{\alpha\beta}(\vec{p}', \vec{p}'') = \frac{C\delta_{\alpha\beta}}{(2\pi)^3}\, e^{-a^2p'^2/4}\,\delta(\vec{p}) \qquad (4.11)$$

which, as $a \to 0$, goes over into (4.10).

Let us now turn to the case of a semi-infinite medium, filling the half-space $x_3 < 0$. If here, too, we preserve with respect to the coordinate (x_3), the expansion of $\vec{K}$ into the complex integral (4.1), then, in the evaluation of $G_{\alpha\beta}$ the integration with respect to x_3' and x_3'' in (4.4) will be now between the limits $-\infty$ and 0. Introducing, for the assurance of convergence of the integrals, a small positive imaginary part $i\gamma$ into $p_3'(\gamma > 0)$ and a similar negative part into p_3'', $G_{\alpha\beta}$ can be obtained in the form of a function having a simple pole at the point $p_3'' = p_3' + 2i\gamma$ in the upper half plane of the complex variable p_3''. However, the artificiality of such a procedure as well as the complication of the calculations with the correlation function $G_{\alpha\beta}$ thus obtained make a choice of another expansion of $\vec{K}$ in the case under study preferable. Namely, it is expedient to use the representation of $\vec{K}$ in the form of a Fourier integral in $\cos p_3x_3$

$$\vec{K}(\vec{r}) = \int_{-\infty}^{+\infty} \vec{g}(\vec{p})\, e^{i(p_1x_1 + p_2x_2)} \cos p_3x_3 \, d\vec{p} \qquad (4.12)$$

Accordingly

$$\vec{g}(\vec{p}) = \frac{2}{(2\pi)^3} \int_{-\infty}^{+\infty} dx_1\, dx_2\, e^{-i(p_1x_1 + p_2x_2)} \int_{-\infty}^{0} \vec{K}(\vec{r}) \cos p_3x_3 \, dx_3 . \qquad (4.13)$$

In what is to follow, in the problem on radiation of an absorbing medium occupying a half-space, we shall be interested only in the pole exterior to the medium. In all such cases we can immediately use the δ-correlation (3.5) for the components of $\vec{K}$

$$F_{\alpha\beta}(r) = \overline{K_\alpha(\vec{r}\,')\, K^*(\vec{r}\,'')} = C\delta_{\alpha\beta}\, \delta(\vec{r}\,' - \vec{r}\,'').$$

Substituting (4.13) and (3.5) into (4.3), we obtain $(\vec{p} = \vec{p}\,' - \vec{p}\,'')$

$$G_{\alpha\beta}(\vec{p}',\vec{p}'') = \frac{4C\delta_{\alpha\beta}}{(2\pi)^6} \int_{-\infty}^{+\infty} dx_1' \, dx_2' \, e^{-i(p_1x_1' + p_2x_2')} \int_{-\infty}^{0} \cos p_3'x_3' \cos p_3''x_3' \, dx_3' .$$

But

$$\int_{-\infty}^{0} \cos\alpha\xi \cos\beta\xi \, d\xi = \frac{1}{2} \int_{-\infty}^{+\infty} \cos\alpha\xi \cos\beta\xi \, d \quad = \frac{\pi}{2}\left\{\delta(\alpha-\beta) + \delta(\alpha+\beta)\right\}.$$

Using this formula and (4.6), we obtain

$$G_{\alpha\beta}(\vec{p}',\vec{p}'') = \frac{C\delta_{\alpha\beta}}{(2\pi)^3}\delta(p_1)\delta(p_2)\left\{\delta(p_3' - p_3'') + \delta(p_3' + p_3'')\right\}.$$

It is not difficult to see that the integral in p_3' and p_3'' of the product of $G_{\alpha\beta}$ by any function $F(p_3', p_3'')$ which is even with respect to each of the arguments, has the same value as the integral evaluated with the help of the correlation function

$$G_{\alpha\beta}(\vec{p}',\vec{p}'') = \frac{2C\delta_{\alpha\beta}}{(2\pi)^3}\delta(\vec{p}). \tag{4.14}$$

Since we shall have to deal only with the functions $F(p_3', p_3'')$ satisfying the indicated condition of evenness, we shall simply use (4.14). In this manner, in the cases of interest to us, the expansion of $\vec{K}$ according to $\cos p_3x_3$ gives for the half-space the same δ-correlation as the expansion according to $e^{ip_3x_3}$ in an unbounded medium, and the only difference with (4.10) lies in the two-fold coefficient.

It now remains for us to determine the meaning of the correlation constant C, which we shall do in Chapter II. For the solution of this problem it is necessary to adduce those expressions for the spectral density of the thermal radiation energy and its intensity, which are known from classical theory. In the following section we shall briefly recapitulate the essential results of this theory.

Section 5. Certain Results From the Classical Theory of Thermal Radiation.

One of the fundamental results, established by Kirchhoff[1] on the basis of the application of thermodynamic laws to equilibrium thermal radiation, was, as is known, the proof that the spectral energy density of this radiation in vacuum u_{ω} is a universal function of frequency and temperature.[2] The structure of the function u_{ω} can only be partially established within the framework of thermodynamics which was subsequently done by Wien.[3] The complete establishment of the form of the universal function constituted a problem for a later stage in the development of the theory of thermal radiation, a stage characterized by the application of statistical methods. W. A. Michelson[4] was the first to draw attention to the fact that it is necessary to adduce molecular-statistical concepts for the theoretical derivation of the energy distribution in the spectrum of equilibrium radiation, and he achieved on this path well known success.[5] Following this idea of W. A. Michelson, Wien went further and obtained for u_{ω} a formula, which represents well the actual energy distribution

1. G. Kirchhoff, Ges. Abh., p. 566, 571; Pogg. Ann. 109, 299, 1860.

2. The initial concept of temperature was applied only to radiating bodies. Wien (see next footnote) notes that Wiedeman (Wied. Ann. 34, 446, 1888) had already pointed to the necessity of introducing the concept of temperature into radiation. In actual fact, Wiedeman introduced only the concept of luminescence temperature, understanding by this the temperature of bodies which give on a given wavelength the equivalent thermal radiation. The expansion of the concept of temperature to the field of radiation is due to B. B. Golitzin ("Investigations in Mathematical Physics", 11, p. 25, 1892).

3. W. Wien, Wied. Ann. 52, 132, 1894.

4. W. A. Michelson, JRFHO (Phys. part) 19, 79, 1887; Journ. de phys. (2), 6, 1887.

5. See W. A. Sokoloff, UFN 43, 275, 1951.

in the region of high frequencies.[1] The final solution of the problem, based on the quantum hypothesis and leading to the expression for u_ω valid for any frequency, was given, as is known, by Planck.[2]

According to Planck's law, the space energy density of equilibrium radiation in vacuum in the interval of (positive) frequencies from ω to $\omega + \Delta\omega$ is[3]

$$u_{0\omega}\Delta\omega = \varepsilon(\omega,T)\cdot\frac{\Delta Z}{V}$$

where $\varepsilon(\omega,T)$ is the mean energy of the oscillator at temperature T

$$\varepsilon(\omega,T) = \frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{\hbar\omega/\theta}-1} \quad ; \quad \theta = kT \tag{5.1}$$

and ΔZ is the asymptotic value of the number of natural oscillations (of the oscillator) of the field in a sufficiently large volume V, lying in the interval of wave-numbers $(k, k + \Delta k)$

$$\Delta Z = V\cdot\frac{k^2\Delta k}{\pi^2} = V\cdot\frac{\omega^2\Delta\omega}{\pi^2 c^3} \tag{5.2}$$

Thus

$$u_{0\omega} = \frac{\hbar\omega^3}{\rho^2 c^3}\left(\frac{1}{2} + \frac{1}{e^{\hbar\omega/\theta}-1}\right). \tag{5.3}$$

In the domain $\hbar\omega \ll \theta$ we have $\varepsilon(\omega,T) = \theta$, i.e., ε is equal to the mean energy, attributable to one degree of freedom according to the theorem of equipartition of classical statistics. Expression (5.3) thus

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1. W. Wien, Wied. Ann. 58, 662, 1896.

2. M. Planck, Wied. Ann. 1, 69, 719, 1900; 4, 553, 1901; 6, 818, 1901; 9, 629, 1902.

3. In what follows we shall denote quantities referring to a vacuum by the subscript 0.

becomes the law of Rayleigh-Jeans[1]

$$w_{0\omega} = \frac{\Theta\omega^2}{\pi^2 c^3} = \frac{\Theta k^2}{\pi^2 c} \tag{5.4}$$

For what is to follow, it is of interest to stress the conditions of validity for (5.2). Let us assume, for the sake of simplicity, that the volume V is a cube of edge ℓ. In the determination of the asymptotic meaning (as $\ell \to \infty$) of ΔZ two conditions must be met:

1) The volume of a spherical layer of thickness Δk in the first octant of the wave number space, equal to $\frac{\pi}{2} k^2 \Delta k$, must be large in comparison to the element volume $(\frac{\pi}{\ell})^3$ of the grid of natural wave numbers. In other words, ΔZ, equal to twice (for the accounting of independent polarizations) the ratio of these volumes, must be sufficiently large

$$\Delta Z = 2 \frac{\frac{\pi}{2} k^2 \Delta k}{(\frac{\pi}{\ell})^3} = \frac{\ell^3 k^2 \Delta k}{\pi^2} \gg 1.$$

2) The correction to ΔZ, of the order of the quantity $\ell^2 \Delta k$, must be small by comparison to the main term. This results in

$$\frac{1}{k\ell} \ll 1.$$

Ignoring the numerical coefficients and expressing both inequalities in terms of wavelength ($k = 2\pi/\lambda$), we obtain

$$\frac{\Delta\lambda}{\lambda} \gg \left(\frac{\lambda}{\ell}\right)^3 \quad ; \quad \frac{\lambda}{\ell} \ll 1 \tag{5.5}$$

1. Rayleigh, Sci. Pap., Vol. IV, 483 (Cambridge, 1903), vol. V, 248 (Cambr. 1912); Phil. Mag. 49, 539, 1900; Nature 72, 54, 243, 1905; J. H. Jeans, Nature 72, 101, 293, 1905; Phil. Mag. 10, 91, 1905; Proc. Roy. Soc. 76, 296, 1905. See also H. A. Lorentz, Nuovo Cimento 16, 5, 1908.

2. M. A. Leontovich, Statistical Physics, Section 20 (M. - L., 1949); R. Courant and D. Hilbert, Methods of Mathematical Physics, vol. I, Chapt. VI, Section 4 (M. - L., 1951).

Thus, the conditions for the validity of (5.2) are first, a not too large monochromaticity of the spectral interval and second, sufficiently large dimensions of the volume under study in comparison to λ. It is understood, that the representation of the volume itself of that part of the spherical layer which lies in the first octant of the wave number space, already assumes in the differential form $\frac{\pi}{2} k^2 \Delta k$ that $\Delta k \ll k$. This means that the non-monochromaticity also has an upper limit.

$$\frac{\Delta\lambda}{\lambda} \ll 1 \,. \tag{5.6}$$

It is not difficult to see that both limitations on the width of the spectral interval, giving (5.5) and (5.6), entrain as a consequence the second condition (5.5) -- the condition of the applicability of the approximation of geometrical optics inside of the volume under consideration.

Concerning the intensity of radiation I_ω, the quantity $I_\omega \cos\theta \cdot d\Omega d\sigma d\omega$ is the energy flow in the interval of positive frequencies from ω to $\omega + d\omega$, passing in unit time through the small surface $d\sigma$ in the solid angle $d\Omega = \sin\theta \, d\theta \, d\varphi$, whose axis forms the angle θ with the normal N to $d\sigma$. Therefore, the power, transformed through the unit surface whose normal is N, is equal to

$$S_{\omega_N} = \int_{\theta \leq \pi/2} I_\omega \cos\theta \, d\Omega = \int_0^{2\pi} d\varphi \int_0^{\pi/2} I_\omega \cos\theta \sin\theta \, d\theta. \tag{5.7}$$

Since, in an isotropic medium, equilibrium radiation, too, is isotropic, i.e., I_ω is independent of direction, we have in this case

$$S_{\omega N} = \pi I_\omega \tag{5.8}$$

In a transparent isotropic medium the intensity I_ω is related to the volume density of electro-magnetic energy u_ω by the expression[1]

1. See, for instance, M. A. Leontovich, Introduction to Thermodynamics, Section 25 (M. - L., 1951).

$$u_\omega = \int \frac{I_\omega d\Omega}{w} \qquad (5.9)$$

where $w = \frac{\partial\omega}{\partial P}$ is the group velocity of frequency ω (P is the wave number in the medium). Thus, for equilibrium radiation

$$u_\omega = \frac{4\pi I_\omega}{w} \qquad (5.10)$$

In particular, for vacuum ($w = c$) it follows from (5.8) and (5.10) that

$$I_{0\omega} = \frac{S_{0\omega N}}{\pi} = \frac{c}{4\pi} u_{0\omega}, \qquad (5.11)$$

i.e., in the region of applicability of the Rayleigh-Jeans law (5.4)

$$I_{0\omega} = \frac{\theta k^2}{4\pi^3}. \qquad (5.12)$$

As is known, one of the Kirchhoff laws establishes the following relation between the equilibrium intensity in the medium I_ω, the emissivity of the medium η_ω and its absorptivity α_ω:

$$I_\omega = \frac{\eta_\omega}{\alpha_\omega}. \qquad (5.13)$$

According to the definition of the emissivity, $\eta_\omega dVd\Omega$ is the power emitted in the frequency interval $(\omega, \omega + d\omega)$ into the solid angle $d\Omega$ from the elementary volume dV of the medium, which is in the interior of the medium itself; α_ω represents merely the energy index of wave extinction in the medium under study.

The relation between the equilibrium value I_ω in a transparent isotropic medium and the intensity of equilibrium radiation in vacuum is established by Clausius' law[1]

1. R. Clausius. Die mechanische Wärmetheorie, vol. I, 335, (Braunschweig, 1887).

$$I_\omega = I_{0\omega} n^2 \qquad (5.14)$$

where n is the index of refraction of the medium for the frequency ω. If there is no scattering, i.e., n is independent of the frequency ω, then $w = c/n$ and from (5.10), (5.11) and (5.14) it follows that

$$u_\omega = u_{0\omega} n^3 . \qquad (5.15)$$

Expression (5.14) gives the volume form of the law relating the intensity in the medium to that in vacuum. For a flat surface, separating a homogeneous medium from vacuum, the radiation intensity of the medium in vacuum is given by Kirchhoff's law

$$I_\omega = I_{0\omega}(1 - R), \qquad (5.16)$$

where R is the energy coefficient of reflection, dependent on the direction of incident waves, on the frequency and on the optical constants of the medium.

For the problems to be studied later in Section 8, it is necessary to define better the meaning of the quantities I_ω and η_ω and the limitations of the applicability of the laws (5.13) - (5.16), of which the first and the last clearly refer to an absorbing medium, while the other two have a definite meaning only for completely transparent media.

First of all, let us note, that the law (5.16) does not contain any limitations on the absorption magnitudes, being a straight consequence of the condition of thermal equilibrium. In fact, at equilibrium, the intensity of the waves incident from vacuum on the plane boundary of the medium at an angle θ is equal to $I_{0\omega}$. The reflected waves have the intensity $I_{0\omega}R$ which, added to I_ω -- the intensity of the waves emitted by the medium in the direction of reflection -- must give $I_{0\omega}$. This is what is expressed by (5.16).

In a transparent medium, radiation is created only by external bodies. In an absorbing medium, each volume element of the medium

itself becomes a source for the chaotic thermal radiation as well. Thus, due to the nature of absorption, the energy flow is conditioned by the wave field of the emitter, as well as by the quasi-stationary field, which is inversely proportional to the second and third power of the distance. The emissivity η_ω characterizes that part of the total energy flow from the element dV of the medium which falls off with the distance from dV only according to the exponential law (with the index α_ω), i.e., that part which is conditioned by the wave field. In order that this fraction of the flow be predominant even at distances r from the emitting element dV of the order of the correlation radius,[1] a sufficiently small extinction is necessary.

Thus, the introduction of η_ω and the law (5.13) are meaningful only under this condition.

Despite the fact that the determination of the intensity (5.7) is evidently not related to the magnitude of absorption, the condition of a small absorptivity is extended to (5.14) as well. The fact is that in all derivations (5.14) the intensity in any solid angle is treated as energy, transferable in a bundle of plane, nonextinguishable waves whose normals are included in this solid angle.[2] But this usual sense included in the understanding of intensity is lost in the presence of absorption, since in this case the field cannot be represented as a combination of plane nonextinguishable waves. Moreover, since each volume element of the absorbing medium constitutes a source of thermal radiation, the latter cannot be presented in the form of a combination of plane extinguishable waves, which constitute particular solutions of homogeneous field equations (1.9) or (1.10) (Section 8). Such waves can only be "admitted" into the absorbing medium from without or can be excited in a corresponding

1. Let us remember that in the case of a zero correlation radius the energy quantities in an absorbing medium are infinite.

2. In a homogeneous isotropic medium the direction of the vector of energy flux coincides with the wave normal.

manner by given (i.e., external) regular sources, distributed in the medium itself. For this reason attempts to generalize (5.14) for the case of strongly absorbing media by studying reflection and refraction of plane waves at the separating boundaries of such media (or absorbing and transparent media) are doomed to failure. Nonetheless, similar attempts had been made.

The first step in such a direction was taken by Laue.[1] He obtained the following result for a medium having a complex index of refraction $n(1 - i\varkappa)$:

$$I_{\omega} = I_{0\omega} n^2 (1 + \varkappa^2)$$

In a recent paper Fragstein[2] more attentively deduced the relation between the coefficients of reflection and transmission at the separating boundary of absorbing media and, having corrected the mistake made at this point by Laue, obtained the formula

$$I_{\omega} = \frac{I_{0\omega} n^2}{1 + \varkappa^2}. \tag{5.17}$$

However neither one, nor the other, formula for the equilibrium intensity within an absorbing medium is compatible with Kirchhoff's law (5.16) for the radiation intensity of such a medium in vacuum.

Using Fresnel's formulae for plane waves with two independent polarizations and the relation between the solid angles in the medium and in vacuum, which are derivable from the refraction law, the following can be shown. If the equilibrium intensity within an unbounded medium is isotropic and is equally distributed between the two polarizations, then the intensity of external radiation of the [illegible]d medium cannot be brought to the form (5.16). The requirement that the intensity (5.16)

1. H. Laue, Wied. Ann. 32, 1085, 1910. See also M. Born and R. Ladenburg, Phys. Zst. 12, 198, 1911.

2. C. Fragstein. Ann. d. Phys. 7, 63, 1950.

take place from without is met only with anisotropic intensity of equilibrium radiation within an unbounded medium, which becomes absurd for a homogeneous and isotropic medium. The mentioned anisotropic intensity in the direction normal to the boundary of the region assumes precisely the meaning (5.17), found by Fragstein, who limited himself precisely to the case of normal incidence. He therefore did not notice that the equilibrium intensity in an unbounded medium thus calculated must be different for other directions, and also different depending on the angle for various polarizations.

The absorption has only a second order effect on the law of refraction and on the coefficients of transmission and reflection.[1] Therefore, if we denote the equilibrium intensity within a medium by $I_{\omega\,\mathrm{equi.}}$, the evaluation of the intensity of external radiation i_ω indicated above leads, for small absorptions, to the formula

$$I_\omega = \frac{I_{\omega\,\mathrm{equil.}}}{n^2}(1 - R)[1 + O(\varkappa^2)]\,. \qquad (5.18)$$

With a precision up to the second order with respect to $\varkappa$, the last multiplier is equal to one, and the entire evaluation of external radiation will be the same as in the case of the boundary region of two transparent media. The coinciding or non-coinciding of (5.18) with Kirchhoff's law (5.16) depends thus on the order of the difference between $I_{\omega\,\mathrm{equil.}}$ and $I_{0\omega}n^2$. The study of this and the other questions touched upon here on the basis of fluctuation theory we shall defer to Section 8. However, from (5.18) it is now clear that second order quantities with respect to $\varkappa$ must in any case be neglected and, consequently, the correction appearing in (5.17) has no meaning.

1. Let us note that in the well-known work of Hilbert (D. Hilbert, Phys. Zst. 13, 1056, 1912) in which the proof of Kirchhoff's laws with the approximations of geometrical optics is studied, it is equivocally admitted that absorptivity does not influence the form of rays. This already introduces the admittance of the possibility of neglecting the absorption with a precision up to the second order.

The refusal to operate with the concept of radiation intensity in strongly absorbing media (whose necessity is usually not clearly discussed) does not lead to any difficulties. Factually this magnitude must serve for the establishment of energy flow through the boundary of the medium and other bodies. The requirement that the solution of this last problem (always existing) under any conditions be formulated in terms of radiation intensity within an emitting medium, is in fact not based on anything.

Up to now we have been considering a homogeneous medium, either unbounded or filling a semi-infinite space, i.e., cases for which the approximations of geometrical optics are valid for waves of any length. For bodies of finite dimensions the applicability of the laws of Kirchhoff -- in their usual form given above -- assumes the possibility of localizing the energy flow in ray-tubes (bundles) and of fixing the flow through limited surfaces. In other words, Kirchhoff's laws are asymptotic in the same sense ($\lambda \ll \ell$) as the laws of Planck or Rayleigh-Jeans. In particular, the law (5.16) assumes that the dimensions of the body and of radii of curvature of its surfaces are very large by comparison to the wavelength. As had been noted in the introduction, the theory developed in the present work is free of this kind of a limitation.

As far as I know, up to the present time the generalizations of the laws (5.14) and (5.15) for the cases of transparent anisotropic and magneto-active media had not been given. To a certain extent, without doubt, this is due to the large complication of the usual derivation based on the consideration of reflection and refraction at the region boundary. In particular, the principle of reciprocity cannot be used in magneto-active media. The fluctuation theory under consideration permits a new approach towards the establishment of the equilibrium energy magnitudes in transparent media with arbitrary anisotropy (Chapter IV). Although the evaluations here, too, are not simple (their unwieldiness is connected with the conditions of the problem itself) the general electrodynamic formulation of the problem renders the whole procedure for solution to a great extent automatic.

CHAPTER II. RADIATION IN A HOMOGENEOUS ISOTROPIC MEDIUM

Section 6. Radiation of a Medium Occupying a Half-space

For the final determination of the correlation function (3.5) we had to find the value of the constant C entering this function. This can be done as follows. Solve the problem of thermal radiation for any concrete case with the aid of the electrodynamic equations (1.10), the boundary conditions (1.13) and the correlation function (3.5). In the solution thus found, one must then go over to the asymptotic approximation, i.e., to high enough frequencies for the classical laws of Kirchhoff to be valid. Direct comparison of the indicated asymptotic approximation with the solution of the same problem obtained directly from Kirchhoff's laws then permits the determination of the value of C.

From the calculational point of view, the simplest problem would be the one concerning the intensity of equilibrium radiation in an unlimited homogeneous medium, and besides, in an absorbing medium, which the formulation itself of the problem of determining C requires. But, as was clarified in the previous section, in order to introduce the intensity within an absorbing medium, we are forced to assume that the absorption is sufficiently small. This kind of a limitation is completely undesirable in the determination of the coefficient in the correlation function, which must remain valid for media with any absorption strength. This forces us to choose another way for the determination of C, in which radiation would not be studied in the interior of the medium but in the non-"noisy" medium.

The simplest in such class of problems is the problem concerning radiation of an absorbing medium, occupying a semi-infinite medium, bounded by a plane surface. Beside calculational advantages we also have here a series of substantial positive moments. It is necessary to compare the result of the solution of the indicated electrodynamic problem directly with Kirchhoff's law (5.16), which, first of all, is not dependent on any assumptions concerning the magnitude of the absorption and, second, is valid under the given geometric conditions

for any wave length. Also it is no longer necessary to go from an exact solution of the fluctuation-electrodynamic problem to an asymptotic approximation for high frequencies.

Let the half-space $z < 1$ (Fig. 1) be filled with a conducting medium. We shall at first assume that the region $z > 0$ is vacuum.

The primary field of the medium, whose potentials we shall denote be $\vec{E}_0$, $\vec{H}_0$, is governed by the external field $\vec{K}$ and satisfies equation (1.10)

$$\operatorname{curl} \vec{E}_0 = -ik\mu\vec{H}_0 \qquad (\mu'' = 0)$$

$$\operatorname{curl} \vec{H}_0 = ik\varepsilon\vec{E}_0 + ik(\varepsilon - 1)\vec{K}$$

In other words, $\vec{E}_0$ and $\vec{H}_0$ represent particular solutions of these non-homogeneous operations.

In accordance with the conception (4.12) for the external field $\vec{K}$ in the semi-space under study

$$\vec{K} = \int_{-\infty}^{+\infty} \vec{g}(\vec{p}) e^{i(p_1x+p_2y)} \cos p_3z \, d\vec{p} =$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty} \vec{g}(\vec{p}) \left(e^{i\vec{p}\cdot\vec{r}} + e^{i\vec{q}\cdot\vec{r}}\right) d\vec{p} \qquad (6.1)$$

where

$$q_1 = p_1 \quad ; \quad q_2 = p_2 \quad ; \quad q_3 = -p_3 \qquad (q^2 = p^2) \qquad (6.2)$$

we shall write the primary field in the form

$$\vec{E}_0 = \int_{-\infty}^{+\infty} \left\{\vec{a}(\vec{p}) e^{i\vec{p}\cdot\vec{r}} + \vec{b}(\vec{q}) e^{i\vec{q}\cdot\vec{r}}\right\} d\vec{p}$$

$$\vec{H}_0 = -\frac{1}{k\mu}\int_{-\infty}^{+\infty} \left\{[\vec{p},\vec{a}] e^{i\vec{p}\cdot\vec{r}} + [\vec{q},\vec{b}] e^{i\vec{q}\cdot\vec{r}}\right\} d\vec{p}. \qquad (6.3)$$

The first basic equation has thus already been utilized, and the substitution from (6.3) into the second basic equation determines $\vec{a}$ and $\vec{b}$ in terms of $\vec{g}$

$$\vec{a} = \frac{\varepsilon - 1}{2\varepsilon} \cdot \frac{k^2 \varepsilon \mu \vec{g} - \vec{p}(\vec{p},\vec{g})}{p^2 - k^2 \varepsilon \mu},$$

$$\vec{b} = \frac{\varepsilon - 1}{2\varepsilon} \cdot \frac{k^2 \varepsilon \mu \vec{g} - \vec{q}(\vec{q},\vec{g})}{p^2 - k^2 \varepsilon \mu} \qquad (6.4)$$

...ide the primary field, the medium also contains a reflected field $\vec{E}_r$, $\vec{H}_r$ which satisfies the uniform field (1.10). For uniform fields we have div $\vec{E}_r = 0$ and, therefore, the field $\vec{E}_r$, $\vec{H}_r$ can be represented in the form of a conjugation of transverse plane waves

$$\vec{E}_r = \int_{-\infty}^{+\infty} \vec{u}\,(p_1,p_2) e^{isr}\, dp_1\, dp_2$$

$$\vec{H}_r = -\frac{1}{k\mu} \int_{-\infty}^{+\infty} [\vec{s},\vec{u}]\, e^{i\vec{s}\vec{r}}\, dp_1\, dp_2 \qquad (6.5)$$

where

$$s_1 = p_1 \quad ; \quad s_2 = p_2 \; ;$$

$$s_3 = +\sqrt{k^2 \varepsilon \mu - (p_1^2 + p_2^2)} \qquad (s^2 = k^2 \varepsilon \mu). \qquad (6.6)$$

An analogous expansion for the field $\vec{E}$, $\vec{H}$ in the half space $z > 0$ (in vacuum) is written in the form

$$\vec{E} = \int_{-\infty}^{+\infty} \vec{v}(p_1,p_2) e^{i\vec{t}\vec{r}}\, dp_1\, dp_2 \qquad (6.7)$$

$$\vec{H} = -\frac{1}{k}\int_{-\infty}^{+\infty} [\vec{t},\vec{v}]e^{i\vec{t}\vec{r}}\,dp_1\,dp_2 \tag{6.7}$$

where

$$t_1 = p_1 \quad ; \quad t_2 = p_2 \;;$$

$$t_3 = -\sqrt{k^2 - (p_1^2 + p_2^2)} \quad ; \quad (t^2 = k^2). \tag{6.8}$$

Thus, all phase indices are equal on the boundary of the region

$$\vec{p}\vec{r} = \vec{q}\vec{r} = \vec{s}\vec{r} = \vec{t}\vec{r} = p_1x + p_2y$$

and therefore the boundary conditions (1.13), taking, in the case under consideration, the form

$$\begin{aligned} [\vec{N},\vec{E}_0 + \vec{E}_r] &= [\vec{N},\vec{E}] \\ [\vec{N},\vec{H}_0 + \vec{H}_r] &= [\vec{N},\vec{H}] \end{aligned} \qquad (z = 0)$$

($\vec{N}$ is the unit vector on the z-axis, i.e., the normal to the boundary) give

$$\int_{-\infty}^{+\infty} [\vec{N},\vec{a} + \vec{b}]\,dp_3 + [\vec{N},\vec{u}] = [\vec{N},\vec{v}],$$

$$\frac{1}{\mu}\int_{-\infty}^{+\infty}\left\{[\vec{N}[\vec{p},\vec{a}]] + [\vec{N}[\vec{q},\vec{b}]]\right\}dp_3 + \frac{1}{\mu}[\vec{N}[\vec{s},\vec{u}]] = [\vec{N}[\vec{t},\vec{v}]]. \tag{6.9}$$

In the components -- this constitutes four equations, to which two more conditions of transversality must be added: for the reflected field (div $\vec{E}_r = 0$) and for the field in vacuum (div $\vec{E} = 0$)

$$\vec{s}\vec{u} = 0 \quad ; \quad \vec{t}\vec{v} = 0. \tag{6.10}$$

We thus have in all six equations which determine $\vec{u}$ and $\vec{v}$ in terms of $\vec{a}$ and $\vec{b}$, i.e., according to (6.4) in terms of $\vec{g}$.

For what is to follow it is sufficient to find only $\vec{v}$, since we are interested in the intensity of radiation I_ω in vacuum ($z > 0$). In this, I_ω can be determined from the expression for the mean density of energy flow

$$S_\omega = \frac{c}{4\pi}\left\{\overline{[\vec{E},\vec{H}^*]} + \overline{[\vec{E}^*,\vec{H}]}\right\}. \tag{6.11}$$

The manner of evaluation is as follows. Express $\vec{v}$ in terms of $\vec{a}$ and $\vec{b}$ from (6.9) and (6.10); then, using (6.4), in terms of $\vec{g}$. Now substitute into (6.11) the expressions $\vec{E}$ and $\vec{H}$, determined by formulae (6.7). After this, one must utilize the correlation function $\vec{g}$ for the half-space, i.e., the formula (4.14)

$$\overline{g_\alpha(\vec{p}\,')\, g^*_\beta(\vec{p}\,'')} = \frac{2C\delta_{\alpha\beta}}{(2\pi)^3}\cdot\delta(\vec{p}\,' - \vec{p}\,''). \tag{6.12}$$

The indicated derivation is given in Appendix I. The following is the result:

$$I_\omega = \frac{k^3 c|\varepsilon - 1|^2 C}{8\pi^3(\varepsilon^* - \varepsilon)}\left\{1 - \frac{1}{2}\left|\frac{\mu\cos\theta - \sqrt{\varepsilon\mu - \sin^2\theta}}{\mu\cos\theta + \sqrt{\varepsilon\mu - \sin^2\theta}}\right|^2 - \frac{1}{2}\left|\frac{\varepsilon\cos\theta - \sqrt{\varepsilon\mu - \sin^2\theta}}{\varepsilon\cos\theta - \sqrt{\varepsilon\mu - \sin^2\theta}}\right|^2\right\} \tag{6.13}$$

The expression in curly brackets is nothing else but

$$1 - R = 1 - \frac{R_\perp + R_\parallel}{2}$$

where $R_\perp$ and $R_\parallel$ are the energy coefficients of reflection from the

medium under study at an angle of incidence θ, referring, respectively, to the polarizations when the electrical vector is perpendicular to the plane of incidence and when it is parallel to it. Since, within the conditions of the problem at hand both polarizations enjoy completely equal rights, the total coefficient is simply equal to the half-mean of $R_{\perp}$ and $R_{\parallel}$. Thus

$$I_{\omega} = \frac{k^3 ci|\varepsilon - 1|^2 C}{8\pi^3(\varepsilon^* - \varepsilon)}(1 - R).$$

But, according to Kirchhoff's law, the non-equilibrium (one-sided) intensity of radiation of a uniformly heated medium, occupying the half-space $z < 0$, is related to the equilibrium intensity $I_{0\omega}$ of radiation in vacuum by relation (5.16)

$$I_{\omega} = I_{0\omega}(1 - R). \tag{6.14}$$

Comparing both expressions for I_{ω}, we obtain the value C

$$C = \frac{8\pi^3(\varepsilon^* - \varepsilon)}{k^3 ci|\varepsilon - 1|^2} I_{0\omega} = \frac{16\pi^3}{k^3 c} I_{0\omega} \operatorname{Im}\left(\frac{1}{\varepsilon - 1}\right). \tag{6.15}$$

In the case of Planck's law of distribution, this gives

$$C = 4\hbar\left(\frac{1}{2} + \frac{1}{e^{\hbar\omega/\theta} - 1}\right) \operatorname{Im}\left(\frac{1}{\varepsilon - 1}\right), \tag{6.16}$$

and in the region of applicability of the Rayleigh-Jeans law, according to (5.11)

$$C = \frac{4\theta}{\omega} \operatorname{Im}\left(\frac{1}{\varepsilon - 1}\right). \tag{6.17}$$

In accordance with (6.15), a medium not having any losses (ε -real) is not a source of radiation ($C = 0$). Of course, the intensity of thermal

radiation in the region $z > 0$ as well as within the medium $z < 0$ can very well be different from zero, inasmuch as the absence of losses means complete transparency of the medium ~~and~~, consequently, the possibility of radiation arrival from infinitely far removed bodies.

Let us note, that had we introduced the external field not into the generalization but into the simple Ohm's law (see Section 1), then its potential $\vec{K}_0$ would have been related to $\vec{K}$ by formula (1.12)

$$\vec{K}_0 = \frac{i(\varepsilon - 1)}{\varepsilon''}\vec{K}.$$

For the correlation functions of the components of $\vec{K}_0$ we would therefore have had

$$\overline{\vec{K}_{0\alpha}\vec{K}^*_{0\beta}} = \frac{|\varepsilon - 1|^2}{\varepsilon''^2}\,\overline{\vec{K}_\alpha \vec{K}^*_\beta} = C_0 \delta_{\alpha\beta}\,\delta(\vec{r})$$

where

$$C_0 = \frac{|\varepsilon - 1|^2}{\varepsilon''^2} C.$$

Introducing here (6.7), we obtain a very simple expression for the correlation constant C_0

$$C_0 = \frac{4\theta}{\omega\varepsilon''} = \frac{\theta}{\pi\sigma}.$$

In contrast to C, the constant C_0 does not become zero when $\varepsilon'' = 0$, however when $\varepsilon'' \neq 0$ it does not depend on ε'. For good conductors, when $\varepsilon'' \gg \varepsilon' - 1$,

$$C \approx C_0 .$$

It is not difficult to generalize the evaluation of radiation to the case when the half-space $z > 0$ is filled with a transparent medium with permeabilities ε_1 and μ_1 (Appendix I). If $\varepsilon_1\mu_1 > 0$, then the expression $I_{0\omega}\varepsilon_1\mu_1(1 - R)$ is obtained for the intensity in this medium,

from which, according to (5.12) it follows that the intensity of equilibrium radiation in a non-absorbing medium having the index of refraction $n = \sqrt{\varepsilon_1 \mu_1}$ is

$$I_\omega = I_{0\omega} n^2. \tag{6.18}$$

As was said, this result refers to the case $n^2 > 0$. If, however, $n^2 < 0$, i.e., the medium has a negative dielectric permeability (which is possible, for instance, in an ionized gas), then the same formulae give

$$I_\omega = 0 . \tag{6.19}$$

The fact that, in the absence of losses in a medium with $\varepsilon_1 < 0$, the intensity of thermal equilibrium radiation is equal to zero, does not, of course, constitute any kind of a paradox. A fluctuation radiation of the medium itself does not exist in view of the absence of losses in it, but the radiation of bodies placed into such a medium experiences complete reflection, i.e., does not create a one-sided flow of electromagnetic energy.

A finite value for I_ω in the half-space $z > 0$, i.e., in the non-"noisy" medium, is obtained, despite the fact that within this medium the δ-correlation for the lateral field $\vec{K}$ is accepted. As explained in Appendix I, this is a result of its own kind of "diffraction smoothing" of the field structure on the plane $z = 0$. As is known, the field in the region $z > 0$ is determined by the condition of radiation and by the field on the plane $z = 0$. The latter is expressible as a double Fourier integral, which is obtained from (6.7) for $z = 0$. Travelling waves in the half-space $z > 0$, which determine the energy flow, are made up only from those harmonic components $e^{i(p_1 x + p_2 y)}$ of the field on the plane $z = 0$, whose period is greater than the wavelength in the half space $z > 0$. The components, however, with periods less than λ give non-uniform standing waves, which become extinct exponentially upon removal from the boundary $z = 0$. In the case under consideration of a transparent external medium these waves do not transfer energy: they automatically

drop out of the expression for the intensity I_ω.

From the viewpoint of electrodynamics, non-uniform standing waves, localized at the surface of a radiating body, constitute nothing else but a quasi-stationary field of elementary radiations (volume elements of radiating medium). The thermal quasi-stationary field does not participate in the creation of energy transfer and, understandably, introduces its share into the energy density u_ω.

The evaluation of u_ω in the half-space $z > 0$ is completely analogous to the evaluation of density of energy flow S_ω and is given in Appendix I. The energy density u_ω is composed of the energy density of travelling waves u_{waves}, constant throughout the half-space $z > 0$ (this is the radiation energy related to the intensity I_ω), and of the energy density of non-uniform standing waves $u_{\omega\,\text{quas}}$. The latter, at large distances from the boundary of the region, is equal to

$$u_{\omega\,\text{quas}} = \frac{u_{0\omega}}{8}\left(\frac{\varepsilon + \mu}{\sqrt{\varepsilon\mu - 1}} - \frac{\varepsilon^* + \mu}{\sqrt{\varepsilon^*\mu - 1}}\right)\frac{1}{(kz)^2} \quad ; \left(kz \gg \left|\frac{\varepsilon}{\sqrt{\varepsilon\mu - 1}}\right|\right).$$

Upon approaching the boundary, the energy density of the quasi-stationary field increases monotonously and on the boundary itself approaches infinity like $1/z^3$

$$u_{\omega\,\text{quas}} = \frac{u_{0\omega}}{8}\,\frac{i(\varepsilon - \varepsilon^*)}{|\varepsilon + 1|^2}\,\frac{1}{(kz)^3}\ .$$

In this fashion the integral of u_ω, taken over any finite volume adjacent to the boundary region, also is divergent. This is the result of the δ-correlation, accepted for the external fluctuation field $\vec{K}$ in an absorbing medium filling the left half-space.

If for $\vec{K}$ we introduce the correlation function with a correlation radius a different from zero, then for distances $z \lesssim a$ the dependence of $u_{\omega\,\text{quas}}$ on z will be changed and on the boundary a very large, but finite value of the order of $1/(ka)^3$ will be obtained.

The thermal quasi-stationary field adjacent to the surface of the radiating body, whose presence is completely understandable from the viewpoint of fluctuation electrodynamics, is, of course, outside the limits of the classical theory of radiation. Naturally the question arises concerning the possibility of an experimental detection of those very large densities of electromagnetic energy, which exist in the layer of the quasi-stationary field. The results given above cannot be directly utilized for the solution of this problem since upon approaching any "test body" to the surface of radiation the structure of the field changes. Therefore, in order to determine what energy the "test body" utilizes (and only this can be fixed in this type of experiment) and how this utilization depends on $u_{\omega\,quas}$, it is necessary to find the external field in the presence of the "test body". In the next section we shall have occasion to obtain a solution to a problem precisely of this nature.

If a radiating medium is bordering on a medium for which $n^2 < 0$, then in this case, too, the absence of an energy flow [result (6.19)] does not mean absence of a field, which however is now entirely reduced to non-uniform standing waves, extinguishing along z (complete reflection). In view of the absence of travelling waves the total energy density $u_\omega = u_{\omega\,quas}$ does not contain a constant (not depending on z) part and upon removal from the region boundary tends to zero.

The simple results given above are obtained only as long as ε_1 and μ_1 are real. If we admit an effective absorption for the second medium ($z > 0$) then the question immediately arises about the determination of the value of intensity within the medium, describable by non-homogeneous field equations. This question we shall defer to Section 8.

If it is assumed that the second medium, which is absorbing, can be described by homogeneous equations (i.e., assume $\vec{K} = 0$ in it), then we come to the limited formulation of the question, valid only for the condition when the thermal self-radiation of the second medium can be ignored (for instance, for the reason that its temperature is considerably lower than that of the first medium). It should be noted that even in such a case the solution is more complicated than for radiation into a

transparent medium, since, because of absorption outside of the medium, a sharp division of the field into travelling and non-uniform standing waves takes place. The energy flow due to radiation of the first medium becomes, for any $z > 0$, finite and the corresponding expressions for the intensity I_ω can be introduced, but this intensity which takes into account as well the energy flow due to non-uniform waves, is no longer related in a simple manner to the coefficients of reflection at the boundary of two absorbing media (Appendix I).

In conclusion, let us note that the method used in this section for determining the radiation intensity in a transparent medium in principle permits the study also of the case when this medium is anisotropic and magnetoactive. However, at the given stage, the solution of this problem would be too difficult since the source of fluctuation radiation is volumetric (half-space $z < 0$), and the radiation itself is anisotropic (one-sided) and depends on the parameters of the radiating medium. Later we shall be able to remove both these conditions and thereby simplify the formulation of the problem to such an extent that its solution becomes, in practice, feasible (Chapter IV).

Section 7. Isotropic Radiation

In order to go from the case studied above of one-sided radiation of the medium, accupying a half space, to isotropic equilibrium radiation one has to encounter thermal radiation of the same temperature, consisting of waves incident on the region boundary from the side of the vacuum and refracted from it with the energy coefficient of refraction R. Such conditions can evidently be obtained in plane space $0 < z < \ell$ between a medium in half-space $z < 0$ and another medium, having the same temperature and occupying the region $z > \ell$. However, an even simpler formulation of the problem can be obtained by placing at a distance ℓ from the medium boundary $z = 0$ a plane, ideal mirror, parallel to this boundary (Fig. 2). In the region $0 < z < \ell$ which we shall for brevity call plane waveguide, equilibrium radiation must then establish itself.

This problem is of interest in a series of relations.

First, it makes it possible to illustrate very clearly the asymptotic character of the law of spectral energy distribution for equilibrium radiation (laws of Planck or Rayleigh-Jeans).

Second, the simplicity of the relations for radiation in a plane waveguide using the indicated asymptotic approximation, in particular the independence of the intensity from the reflection coefficient of the radiating medium, opens up a direct path for the determination of Kirchhoff's laws for the case when the waveguide is filled with a transparent medium. This problem has in essence already been solved in Section 6 without the waveguide but for an isotropic medium. We still have to clarify what form the Kirchhoff laws assume in anisotropic and magnetoactive media. In this rather general case it is particularly expedient to start with the conditions for which the reflection coefficient is not important. Thus, the contents of this section will also serve as an introduction to Chapter IV.

Finally, if it be assumed that the mirror which is parallel to the surface of the radiating medium is not completely ideal (has a large, but finite conductivity), then it can be used in the capacity of a "test body" about which we spoke in the last section. In other words, the study of the energy attenuated by the mirror permits, albeit on a theoretical level, an answer to the question concerning the detectability of the thermal quasi-stationary field near the surface of the radiating medium.

And so, let the conditions of the problem studied in the previous section be now supplemented by the presence of an ideal plane mirror placed at the surface $z = \ell$. The expressions (6.1) - (6.6) for fields within the medium remain in force and only the form of the field $\vec{E}$, $\vec{H}$ in vacuum need be changed. This field will now be made up not only of waves propagating in the positive direction z, but also of counter-waves, incident on the medium from the side of the mirror.

$$\vec{E} = \int_{-\infty}^{+\infty} \left\{ \vec{v}(p_1, p_2) e^{i\vec{t}\cdot\vec{r}} + \vec{u}(p_1, p_2) e^{i\vec{t}'\cdot\vec{r}} \right\} dp_1 \, dp_2 \qquad (7.1)$$

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$$\vec{H} = -\frac{1}{k}\int_{-\infty}^{+\infty}\left\{[\vec{t},\vec{v}]e^{i\vec{t}\cdot\vec{r}} + [\vec{t}',\vec{w}]e^{i\vec{t}'\cdot\vec{r}}\right\} dp_1\, dp_2 \qquad (7.1)$$

where

$$t_1 = t_1' = p_1 \quad ; \quad t_2 = t_2' = p_2$$

$$t_3 = -t_3' = -\sqrt{k^2 - (p_1^2 + p_2^2)} \qquad (t^2 = t'^2 = k^2), \qquad (7.2)$$

Beside the boundary conditions at the plane $z = 0$ which take the form

$$\int_{-\infty}^{+\infty} [\vec{N}, \vec{a} + \vec{b}]\, d\rho_3 + [\vec{N}, \vec{u}] = [\vec{N}, \vec{v} + \vec{w}]$$

$$\frac{1}{\mu}\int_{-\infty}^{+\infty}\left\{[\vec{N}[\vec{p},\vec{a}]] + [\vec{N}[\vec{q},\vec{b}]]\right\}dp_3 + \frac{1}{\mu}[\vec{N},[\vec{s},\vec{u}]] = [\vec{N}[\vec{t},\vec{v}]] + [\vec{N}[\vec{t}',\vec{w}]] \quad (7.3)$$

we must in addition impose boundary conditions at the plane of the ideal mirror $z = \ell$

$$[\vec{N},\vec{E}] = 0$$

which, as can readily be seen, gives

$$[\vec{N},\vec{v}]e^{it_3\ell} + (\vec{N},\vec{w})e^{it_3'\ell} = 0\,. \qquad (7.4)$$

For the components we thus have four equations (7.3) and two equations (7.4), to which will be added three conditions of transversality of waves

$$\vec{s}\vec{u} = 0 \quad ; \quad \vec{t}\vec{v} = 0 \quad ; \quad \vec{t}'\vec{w} = 0. \qquad (7.5)$$

In sum, we dispose of nine equations which determine $\vec{u}$, $\vec{v}$ and $\vec{w}$ in terms of $\vec{a}$ and $\vec{b}$, i.e., according to (6.4), in terms of $\vec{g}$.

Although all evaluations basically repeat what had been done in Appendix I, for completeness of the presentation we are showing the evaluations in broad outline in Appendix II. For the intensity, computed

for waves which propagate from the medium to the mirror [first terms in flower brackets in (7.1)], we obtain the following expression

$$I_\omega = I_{0\omega}\frac{(\varepsilon^* - \varepsilon)\cos\theta}{4(\zeta^* - \zeta)}\left\{\frac{\mu^2}{|\mu\cos\theta\cos\xi + i\zeta\sin\xi|^2} + \frac{\sin^2\theta + |\zeta|^2}{|i\varepsilon\cos\theta\sin\xi + \zeta\cos\xi|^2}\right\} \tag{7.6}$$

where

$$\zeta = \sqrt{\varepsilon\mu - \sin^2\theta} \quad ; \quad \xi = k\ell\cos\theta. \tag{7.7}$$

In accordance with (7.6), the intensity depends in an oscillatory manner on the distance to the mirror ℓ, while the period (along ℓ) of these interference oscillations of I_ω is equal to $\frac{2\pi}{k\cos\theta} = \frac{\lambda}{\cos\theta}$, i.e., it is larger, the nearer the direction under consideration is to slipping. At fixed ℓ, the oscillations of I_ω take place under an angle θ, the oftener the larger ℓ is. If $\ell \to \infty$, then all these interference phenomena will be completely obliterated already at a very small non-chromaticity $\Delta k = -\frac{2\pi\Delta\lambda}{\lambda^2}$. As ℓ increases without limit, the least deviations from an ideal plane of the surfaces of the mirror or the medium will cause an averaging of the interference picture. All these conditions are completely analogous to those which make it, in practice, impossible to achieve interference phenomena in thick plates.

The condition for the disappearance of interference in radiation, going under the angle θ to the normal, will be, evidently, the presence of a non-chromaticity Δk such that

$$\Delta k \cdot \ell\cos\theta \gg 2\pi, \qquad \frac{\Delta\lambda}{\lambda} \gg \frac{\lambda}{\ell\cos\theta}. \tag{7.8}$$

It can be said that an excessively monochromatic spectral apparatus, having "transmission lines" $\Delta\lambda$ which do not satisfy condition (7.8), will register interference oscillations of the intensity, i.e., will make evident the deviations from Planck's law (in the sense of space deviations from uniformity and isotropicity of the radiation). Understandably, the classical radiation theory does not take into account this type of a

deviation.

At fixed k, the angular width $\Delta\theta$ of the interference band must be very small compared to $\pi/2$, in order that the picture may approach uniformity and isotropicity. The angular width of the band is determined by the relation $|\Delta(k\ell\cos\theta)| = k\ell\sin\theta\cdot\Delta\theta = \pi$. The condition $\Delta\theta \gg \pi/2$ therefore means

$$\frac{\lambda}{\ell} \ll \pi\sin\theta \;. \qquad (7.9)$$

It is not difficult to discover a certain parallelism between conditions (7.8) and (7.9), on the one hand, and on the other hand the conditions of validity of equi-partition, i.e., the conditions (5.5) and (5.6). A complete coinciding cannot exist because conditions (5.5) and (5.6) were obtained for a volume limited on all sides while in the case under consideration we are dealing with a plane layer, unlimited in the x and y directions.

If conditions (7.8) and (7.9) are satisfied, then factually the observable intensity will be equal to the mean of (7.6) with respect to ξ. This averaging can be interpreted as an averaging with respect to the angle θ (with respect to the width of the interference band) as well as an averaging with respect to the wavelength interval $\Delta\lambda$. Using the following formula

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{d\xi}{|a\cos\xi + b\sin\xi|^2} = \frac{2}{|i(ab^* - a^*b)|} \;, \qquad (7.10)$$

we obtain the following averaged values of the first and second terms inside the large brackets of (7.6):

$$\frac{2\mu}{(\zeta^* + \zeta)\cos\theta} \quad \text{and} \quad \frac{2(\sin^2\theta + |\zeta|^2)}{(\varepsilon\zeta^* + \varepsilon^*\zeta)\cos\theta} \;.$$

Substituting these values into (7.6) and carrying out a simple rearrange-

ment, we obtain

$$I_\omega = I_{0\omega},$$

i.e., the expression for the intensity according to (5.11) corresponding to Planck's law or the Rayleigh-Jeans law.

Let us now suppose that the mirror has a finite conductivity, and let us turn to the question concerning the determination of the quasi-stationary field due to the energy attenuated by the mirror.

If the conductivity of the mirror is high, then, first, we need not take into account its own radiation (we can, besides, consider that the temperature of the mirror is maintained sufficiently low), and, second, we can as a first approximation neglect the difference between the magnetic field and the field which exists in the case of an ideal mirror. In other words, to evaluate the energy utilized by the mirror we can use instead of (7.4) the boundary conditions of M. A. Leontovich (see Section 11)

$$H_1 = -\sqrt{\varepsilon_1}\, E_2 \quad ; \quad H_2 = \sqrt{\varepsilon_1}\, E_1 \qquad (z = \ell) \tag{7.11}$$

where ε_1 is the dielectric permittivity of the mirror, related to its conductivity σ by the relation

$$\varepsilon_1 \approx -i\varepsilon_1'' = -i\frac{4\pi\sigma}{\omega} \qquad (\varepsilon_1'' \gg 1). \tag{7.12}$$

The energy flow density into the mirror is

$$S \equiv (S_{\omega 3})_{z=\ell} = \frac{c}{4\pi}\left(\overline{E_1 H_2^*} - \overline{E_2 H_1^*} + \overline{E_1^* H_2} - \overline{E_2^* H_1}\right)$$

or, because of (7.11) and (7.12)

$$S = \frac{c}{4\pi}\sqrt{\frac{\omega}{2\pi\sigma}}\left(\overline{|H_1|^2} + \overline{|H_2|^2}\right), \tag{7.13}$$

where H_1 and H_2 are the tangential component values of the unperturbed magnetic field (field in the presence of an ideally conducting mirror), which we already know. The corresponding evaluations, consisting of substituting H_1 and H_2 from (7.1) into (7.13) and of carrying out statistical averaging, are done in Appendix II and give the following result:

$$S = \frac{\pi I_{0\omega}}{\mu}\sqrt{\frac{\omega}{2\pi\sigma}}\left\{\int_0^{\pi/2}\left(\frac{\mu^2\cos^2\theta}{|\mu\cos\theta\cos\xi + ia\sin\xi|^2} + \frac{\sin^2\theta + |a|^2}{|i\varepsilon\cos\theta\sin\xi + a\cos\xi|^2}\right)(a + a^*)\cos\theta\sin\theta\, d\theta + \right. \tag{7.14}$$

$$+\int_0^{\infty}\frac{\mu^2\sinh^2\psi}{|i\mu\sinh\psi\cosh\eta - b\sinh\eta|^2} + \frac{\cosh^2\psi + |b|^2}{|i\varepsilon\sinh\psi\sinh\eta - b\cosh\eta|^2}\cdot$$

$$\cdot (b + b^*)\cosh \ldots \sinh \quad d \quad .$$

Here

$$\xi = k\ell\cos\theta, \qquad a = \sqrt{\varepsilon\mu - \sin^2\theta},$$
$$\eta = k\ell\sinh\psi; \qquad b = \sqrt{\varepsilon\mu - \cosh^2\psi}. \tag{7.15}$$

The first integral in (7.14) expresses the power used by the mirror, caused by radiation (travelling waves), and the second integral expresses the power caused by the quasi-stationary field (non-uniform waves). Thus, without any quantitative estimations, we can observe that the quasi-stationary field contributes a definite share to the energy attenuated by the mirror. A complete clarification of the behavior of both integrals in (7.14), which for brevity we shall denote by I_1 and I_2, requires numerical integration. But in the limiting cases of very large and very small $k\ell$ it is not difficult to gi.. approximate estimates.

The integral I_1 is finite for all ℓ, including the surface of the radiating medium, at which it assumes the following value

$$I_1 = \frac{4}{3}\left\{(\varepsilon\mu)^{3/2} - (\varepsilon\mu - \frac{1}{4})\sqrt{\varepsilon\mu - 1} + \text{conjugate}\right\} \qquad (\ell = 0).$$

If the medium has large losses, this expression becomes:

$$I_1 = \sqrt{\varepsilon\mu} + \sqrt{\varepsilon^*\mu} = \sqrt{2\varepsilon''\mu} \qquad (\ell = 0,\ \varepsilon'' \gg 1) \quad (7.16)$$

For large distances ($k\ell \gg 1$) between the mirror and the medium $\sin\xi$ and $\cos\xi$ under the integral sign in I_1 oscillate rapidly with θ ($\xi = k\ell\cos\theta$). The value of I_1 can in this case be obtained by preliminary averaging of the integrand with respect to ξ, using formula (7.10). This gives

$$I_1 = \frac{8\mu}{3} \qquad (k\ell \gg 1).$$

Thus, for large ℓ, the wave field causes a constant (independent of either ℓ or of the parameters of the radiating medium) energy flow into the mirror, equal to

$$S_1 = \frac{\pi I_{0\omega}}{\mu}\sqrt{\frac{\omega}{2\pi\sigma}}\, I_1 = \frac{8\pi I_{0\omega}}{3}\sqrt{\frac{\omega}{2\pi\sigma}} \qquad (k\ell \gg 1).$$

The integral I_2 behaves quite differently. As can readily be seen, I_2 approaches 0 as $k\ell$ increases and increases without limit as $\ell \to 0$. Introducing in place of ψ the integration variable η and using for small $k\ell$ the approximate expression

$$b = \sqrt{\epsilon\mu - \cosh^2\psi} \approx \frac{1}{k\ell}\sqrt{k^2\ell^2(\epsilon\mu - 1) - \eta^2} \approx -\frac{1}{k\ell}\left[i\eta + \frac{k^2\ell^2(\epsilon\mu - 1)}{2i\eta}\right],$$

we obtain

$$I^2 \cdot = \frac{(\varepsilon^* - \varepsilon)\mu}{2ik\ell} \int_0^\infty \left(\frac{\mu^2}{(\mu\cosh\eta + \sinh\eta)^2} + \frac{2}{|\varepsilon\sinh\eta + \cosh\eta|^2} \right) d\eta =$$

$$= \frac{\varepsilon''\mu}{k\ell}\left(\frac{\mu}{\mu + 1} + \frac{2}{\varepsilon''}\operatorname{arctg}\frac{\varepsilon''}{\varepsilon' + 1}\right) \qquad (k\ell \ll 1) .$$

In the case of large losses in the emitting medium, this gives

$$I_2 = \frac{\varepsilon''\mu^2}{(\mu + 1)k\ell} \qquad (k\ell \ll 1, \ \varepsilon'' \gg 1) . \tag{7.17}$$

Consequently, the ratio of powers attenuated by the mirror from the quasi-stationary and from the wave fields, for $k\ell \ll 1$ and $\varepsilon'' \gg 1$, is equal, according to (7.16) and (7.17), to

$$\frac{I_2}{I_1} = \frac{\mu}{\mu + 1}\sqrt{\frac{\mu\varepsilon''}{2}}\frac{1}{k\ell} ,$$

i.e., it can be very large. In principle this result confirms the fact that under known conditions the absorbing "best body" can manifest a thermal quasi-stationary field, localized in the layer close to the radiating surface. It is necessary, however, to emphasize that the results obtained here do not refer to the total but to the spectral intensity of this field.

Section 8. Intensity and Density of Energy in an Absorbing Medium

As was noted in Section 3, it can be forseen, merely on the basis

of general considerations, that the expressions for energy (average bilinear) magnitudes within an absorbing medium can be finite only for a correlation radius of the lateral fluctuation field $\vec{K}$ different from zero. Therefore, in initiating our study of the problem concerning the mentioned magnitudes, we shall at first utilize the most general form of the correlation function with the aim of scrutinizing the role of any particular assumptions. The study can be undertaken in two ways: either by applying the expansion of the potentials into Fourier integrals or without these expansions -- by direct integration of the field equation (1.10)

$$\operatorname{curl} \vec{E} = -ik\mu\vec{H}$$

$$\operatorname{curl} \vec{H} = ik\varepsilon\vec{E} + ik(\varepsilon-1)\vec{K}, \qquad (8.1)$$

Initially, we shall use the first method, since the utilization of the general correlation function renders the evaluations by this method easier.

In accordance with the expansion (4.1) for $\vec{K}$ in case of an unbounded medium

$$\vec{K}(\vec{r}) = \int_{-\infty}^{+\infty} \vec{g}(\vec{p})\, e^{i\vec{p}\cdot\vec{r}}\, d\vec{p} \qquad (8.2)$$

we let

$$\vec{E} = \int_{-\infty}^{+\infty} \vec{a}(\vec{p})\, e^{i\vec{p}\cdot\vec{r}}\, d\vec{p} \quad ; \quad \vec{H} = -\frac{1}{k\mu}\int_{-\infty}^{+\infty} [\vec{p},\vec{a}]\, e^{i\vec{p}\cdot\vec{r}}\, d\vec{p} \qquad (8.3)$$

The first of the equations (8.1) is thereby already satisfied, and substitution of (8.2) and (8.3) into the second equation gives

$$\vec{a} = \frac{\varepsilon-1}{\varepsilon}\,\frac{k^2\varepsilon\mu\vec{g} - \vec{p}(\vec{p},\vec{g})}{p^2 - k^3\varepsilon\mu}. \qquad (8.4)$$

Using the general expression (4.8) for the correlation functions of component $\vec{g}$

$$g_{\alpha}(\vec{p})g^{*}_{\beta}(\vec{p}') = \frac{\delta(\vec{p}-\vec{p}')}{(2\pi)^3}\left\{U(p)\delta_{\alpha\beta} - V(p)\frac{p_{\alpha}p_{\beta}}{p^2}\right\} \quad (8.5)$$

and formula (8.4), it is not difficult to obtain the correlation function for a

$$\overline{a_{\alpha}(\vec{p})a^{*}_{\beta}(\vec{p}')} = \frac{|\varepsilon-1|^2\delta(\vec{p}-\vec{p}')}{(2\pi)^3|\varepsilon|^2}\times$$

$$\times\left\{U(p)\frac{k^4|\varepsilon|^2\mu^2\delta_{\alpha\beta} - p_{\alpha}p_{\beta}[k^2(\varepsilon+\varepsilon^{*})\mu - p^2]}{|p^2-k^2\varepsilon\mu|^2} - V(p)\frac{p_{\alpha}p_{\beta}}{p^2}\right\}, \quad (8.6)$$

wherefrom, in particular, it follows that

$$\overline{aa^{*}} = \frac{|\varepsilon-1|^2\delta(\vec{p}-\vec{p}')}{(2\pi)^3|\varepsilon|^2}\left\{\frac{2k^4|\varepsilon|^2\mu^2}{|p^2-k^2\varepsilon\mu|^2}U(p) + U(p) - V(p)\right\}. \quad (8.7)$$

Once again, the expression (2.10) for the density of energy flow will serve us for the determination of the radiation intensity

$$S_{\omega} = \frac{c}{4\pi}\left\{\overline{[\vec{E},\vec{H}^{*}]} + \overline{[\vec{E}^{*},\vec{H}]}\right\}. \quad (8.8)$$

For the density of electro-magnetic energy

$$u_{\omega} = u_{e\omega} + u_{m\omega} \quad (8.9)$$

in the absence of scattering and for real permittivity μ, we have [see (2.11)][1]

1. The assumption of an absence of scattering in the presence of an absorbing medium is, of course, highly specialized, but for the questions under consideration it is not essential.

$$u_{e\omega} = \frac{\varepsilon + \varepsilon^*}{8\pi} \overline{\vec{E}\vec{E}^*}, \qquad u_{m\omega} = \frac{\mu}{4\pi} \overline{\vec{H}\vec{H}^*}. \qquad (8.10)$$

With the help of these expressions, the expansion (8.3) and the correlation function (8.6), we obtain (see Appendix III)

$$I_\omega = \frac{2k^3 c\mu |\varepsilon - 1|^2}{(2\pi)^4} \int_0^\infty \frac{U(p) p^3 dp}{|p^2 - k^2 \varepsilon \mu|^2}$$

$$u_{e\omega} = \frac{k^4 (\varepsilon^* + \varepsilon)\mu^2 |\varepsilon - 1|^2}{(2\pi)^3} \left\{ \int_0^\infty \frac{U(p) p^2 dp}{|p^2 - k^2 \varepsilon \mu|^2} + \right.$$

$$\left. + \frac{1}{2k^4 |\varepsilon|^2 \mu^2} \int_0^\infty [U(p) - V(p)] p^2 dp \right\}, \qquad (8.11)$$

$$u_{m\omega} = \frac{2k^2 \mu |\varepsilon - 1|^2}{(2\pi)^3} \int_0^\infty \frac{U(p) p^4 dp}{|p^2 - k^2 \varepsilon \mu|^2}.$$

Thus, I_ω and $u_{m\omega}$ do not depend on $V(p)$, i.e., in accordance with (3.3) and (4.9), they do not depend on the second (potential) term of the correlation function component of the lateral field $\vec{K}$. The assumption concerning the δ-correlation of $\vec{K}$, which means that

$$U(p) = C, \qquad V(p) = 0,$$

as can easily be seen, does in fact entrain the divergence of all the expressions (8.11). Let us note that the choice $V(p) = U(p) = C$, which does not change anything in I_ω and $u_{m\omega}$, assures the finiteness of electrical energy and gives for its expression

$$u_{e\omega} = \frac{k^3 i(\varepsilon^* + \varepsilon)\mu^{3/2} |\varepsilon - 1|^2 C}{16\pi^2 (\sqrt{\varepsilon^*} - \sqrt{\varepsilon})}.$$

Substituting here the value (6 15) for C

$$C = \frac{9(3, \ldots)}{k^3 c i |\varepsilon - 1|^2} I_{0\ldots} = \frac{2^{2}(\ldots)}{k^3 i|\varepsilon - 1|^2} u_{0} ,$$

we obtain

$$u_{e\omega} = \frac{u_{0\omega}}{2} \quad \frac{\varepsilon^* + \varepsilon}{2} \quad \frac{\sqrt{\varepsilon^*} + \sqrt{\varepsilon}}{2} = \frac{u_{0\omega}}{2} n^3(1 - \ldots^2), \qquad (8.12)$$

where the real and imaginary parts of the complex index of refraction have been introduced in the usual manner

$$\sqrt{\varepsilon} = n(1 - i \ldots) . \qquad (8.13)$$

It can be shown that the choice $V(p) = U(p)$ signifies imposing the condition div $\bar{K} = 0$ and leads to the neglect of longitudinal waves (see Appendix III). Retaining only transverse waves, when discussing the density of radiation energy, may prove to be justifiable. Nevertheless, one must observe that this assumption, essentially completely arbitrary, does not even lead to a physical understanding of the results since the expressions for I_{ω} and u_{ω} for $U(p) = C$ remain divergent.

Let us now see, to what the choice of the correlation function with a correlation radius different from zero, for instance the function (4 11), leads

$$U(p) = Ce^{-a^2p^2/4} , \quad V(p) = 0 \qquad (8\ 14)$$

the evaluation of the integrals in (8 11) is done in Appendix III and gives in this case

$$I_{\omega} = I_{0\omega} n^2 (1 - \ldots^2)\left(1 - \frac{1}{\ldots} \operatorname{arcctg} \frac{1 - \ldots^2}{2 \ldots}\right) +$$

$$+ \frac{4\varkappa}{\pi} \ln(\gamma \, kan\sqrt{1 + \varkappa^2}) + O(\varkappa, a^2)\Big\} ,$$

$$u_{e\omega} = \frac{u_{0\omega}}{2} n^3(1 - \varkappa^2)\left\{1 + \frac{4\varkappa}{\sqrt{\pi}(kan)^3(1 + \varkappa^2)^2} + O(\varkappa, a)\right\} , \quad (8.15)$$

$$u_{m\omega} = \frac{u_{0\omega}}{2} n^3 \Big\{1 - 3\varkappa^2 + \frac{4\varkappa}{\sqrt{\pi}\, kan} + O(\varkappa, a) ,$$

where $\ln\gamma = 0.577216$ is the Euler constant and where $O(\varkappa, a^2)$ denotes, as usual, terms of order not less than $\varkappa$ or a^2. When $a \neq 0$ and $\varkappa \to 0$ these formulae become the law for transparent media

$$I_\omega = I_{0\omega} n^2, \quad u_{e\omega} = u_{m\omega} = \frac{u_{0\omega}}{2} n^3 \quad (8.16)$$

(the latter -- in the absence of scattering).

Let us first of all note that we do not have any physically justified criterion for an unequivocal separation of expression (8.15) into parts, one of which would refer to the internal energy of the medium and the other to the thermal radiation in this medium Of course, the terms containing the correlation radius a, i.e., the terms dependent on the microstructure of the medium, must be attributed to the internal energy, but this does not mean that some of the terms not containing a do not belong to the internal energy. On the contrary, from the fact that only the parts of $u_{e\omega}$ and $u_{m\omega}$ containing a have the first order with respect to $\varkappa$ it becomes quite natural to attribute in the expression for the intensity I_ω the term with arcctg (of the first order with respect to $\varkappa$ for small $\varkappa$) also to the internal energy Then the formulae (8.16), among which the first guarantees the fulfillment of Kirchhoff's law for external radiation (section 5), will be valid with a precision up to the the second order with respect to $\varkappa$. With the same precision the general expression for u

we obtain from (8.1) the equation for $\vec{A}$

$$\nabla^2\vec{A} + q^2\vec{A} = - ik(\varepsilon - 1)\vec{K}, \tag{8.20}$$

where

$$q = k\sqrt{\varepsilon\mu} = k\mu(1 - i\kappa) . \tag{8.21}$$

The solution of equation (8.20), which expresses the retarding potential in the case under study of an unlimited medium not containing any except volume sources, is

$$\vec{A} = \frac{ik(\varepsilon - 1)}{4\pi}\int \frac{\vec{K}e^{-iqr}}{r}\,dV . \tag{8.22}$$

The utilization of potentials is particularly convenient because space differentiation under the sign of the volume integral does not touch $\vec{K}$ and thereby the intensities $\vec{E}$ and $\vec{H}$ in the same manner do not contain derivatives of $\vec{K}$.

Substituting (8.19) and (8.22) into (8.18), we obtain

$$\vec{E} = \frac{\varepsilon - 1}{4\pi\varepsilon}\int\left\{\vec{K}\left(\frac{q^2}{r} - \frac{iq}{r^2} - \frac{1}{r^3}\right) - \vec{r}(\vec{r}, \vec{K})\left(\frac{q^2}{r^3} - \frac{3iq}{r^4} - \frac{3}{r^5}\right)\right\} e^{-iqr}\,dV,$$

$$\vec{H} = -\frac{ik(\varepsilon - 1)}{4\pi}\int [\vec{r}, \vec{K}]\left(\frac{1}{r^3} + \frac{iq}{r^2}\right) e^{-iqr}\,dV . \tag{8.23}$$

Every volume element dV of the medium gives at the point of observation the same kind of field as an electric dipole with the moment $d\vec{p} = \frac{(\varepsilon - 1)\vec{K}}{4\pi}\,dV$; $\vec{E}$ and $\vec{H}$ represent the summed intensities of all such random element fields.

In order to find the energy intensity and density one must now substitute (8.23) into the expressions (8.8) and (8.10) and utilize the

correlation function (8.17).

If we assume the δ-correlation (3.5) for the lateral field $\vec{K}$, i.e., if we let $f(R) = \delta(\vec{r} - \vec{r}')$, then, as shown in Appendix III, the following expressions for the energy quantities of interest to us are obtained:

$$I_\omega = \frac{kc|\varepsilon - 1|^2 c}{32\pi^3|\varepsilon|^2} \int_0^\infty e^{i(q^*-q)r} \left\{ |q|^2(q\varepsilon^* + q^*\varepsilon) - \right.$$

$$\left. - \frac{\varepsilon^* - \varepsilon}{r}\left[i|q|^2 + \frac{q^* - q}{r} + \frac{i}{r^2}\right]\right\} dr,$$

$$u_{e\omega} = \frac{(\varepsilon^* + \varepsilon)|\varepsilon - 1|^2 c}{16\pi^2|\varepsilon|^2} \int_0^\infty e^{i(q^*-q)r} \left\{ |q|^4 - \frac{i|q|^2(q^* - q)}{r} + \right.$$

$$\left. + \frac{3|q|^2 - q^2 - q^{*2}}{r^2} - \frac{3i(q^* - q)}{r^3} + \frac{3}{r^4} \right\} dr , \qquad (8.24)$$

$$u_{m\omega} = \frac{k^2\mu|\varepsilon - 1|^2 c}{8\pi^2} \int_0^\infty e^{i(q^*-q)r} \left\{ |q|^2 - \frac{i(q^* - q)}{r} + \frac{1}{r^2} \right\} dr .$$

All three integrals are divergent which, of course, must take place for the δ-correlation components of $\vec{K}$. The terms which contain r in the denominator and which cause the divergence of the integrals, are obtained because of those parts in the integrands (8.23), which with increasing r decrease faster than e^{-iqr}/r, i.e., they correspond to the quasi-stationary field parts of the elementary dipoles. The wave terms in (8.23), decreasing with increasing r as e^{-iqr}/r, give in (8.24) only the first parts of the integrands. These parts do not contain r in the denominator and lead to the following finite values:

$$I_\omega = -\frac{kc|\varepsilon - 1|^2 c}{32\pi^3|\varepsilon|^2} \cdot \frac{|q|^2(q\varepsilon^* + q^*\varepsilon)}{i(q^* - q)},$$

$$u_{e\omega} = -\frac{(\varepsilon^* + \varepsilon)|\varepsilon - 1|^2 C}{16\pi^2|\varepsilon|^2} \cdot \frac{|q|^4}{i(q^* - q)}, \tag{8.25}$$

$$u_{m\omega} = -\frac{k^2\mu|\varepsilon - 1|^2 C}{8\pi^2} \cdot \frac{|q|^2}{i(q^* - q)}.$$

It appears to be sufficiently probable that to obtain the quantities characterizing the radiation it is necessary to take into account in $\vec{E}$ and $\vec{H}$ the wave terms and only the wave terms. Of course, the separation of the field into wave and quasi-stationary parts in the absorbing medium becomes to a certain extent indefinite. If the dipole is in a transparent medium, then the quasi-stationary terms automatically drop out of the expression for energy flow. This can readily be seen from the first formula (8.24): when $\varepsilon^* = \varepsilon$, the terms with r in the denominators disappear. But in an absorbing medium the quasi-stationary part of the field also participates in creating the energy flow and the neglect of this part in fact appears to be arbitrary.

Let us nevertheless see what the expressions (8.25) obtained from wave fields of elementary random dipoles mean. Introducing into (8.25) the above adduced expression for C (appropriately in terms of $I_{0\omega}$ for the first formula (8.25) and in terms of $u_{0\omega}$ for the other two) and using (8.21), it is not difficult to transform formula (8.25) to the form

$$I_\omega = I_{0\omega} n^2, \qquad u_{e\omega} = \frac{u_{0\omega}}{2} n^3(1 - \varkappa^2),$$

$$u_{m\omega} = \frac{u_{0\omega}}{2} n^3(1 + \varkappa^2), \qquad u_\omega = u_{e\omega} + u_{m\omega} = u_{0\omega} n^3. \tag{8.26}$$

Thus, with a precision up to the second order in $\varkappa$, we obtain the same formulae as for a transparent (non-scattering) medium.

It is interesting to note that the density of electrical energy $u_{e\omega}$

in (8.26) is the same as in (8.12), although both expressions have been obtained under completely different assumptions.

A more consequential path for the separation of energy quantities, which characterize radiation in absorbing media, of course consists not of neglecting the divergent expressions, conditioned by the δ-correlation of K, but of examining those finite quantities which are obtained with a correlation radius different from zero, as was done above in the evaluation using the Fourier expansion.

As an example let us take the following exploded "model" of the correlation function:

$$f(R) = \begin{cases} \frac{3}{4}\pi a^3 & \text{where } R \leq a, \\ 0 & \text{where } R > a. \end{cases} \tag{8.27}$$

With this correlation function,[1] the intensity, calculable with the aid of the full expressions for intensities (8.23), is as follows (see Appendix III):

$$I_{\nu} = \frac{I_{0}}{(1 + \varkappa^2)^2}\left\{\frac{6\varkappa^2}{(ka)^2} + n^2\left(1 - \frac{35\varkappa^4}{3}\right) + \right.$$

$$\left. + 8\varkappa^4 n^2 \ln(2\gamma kna\varkappa) + \zeta(\varkappa, a)\right\}. \tag{8.28}$$

In this manner, in contrast to (8.15), the term which is independent of the correlation radius a is different in this case from expression (8.16) only in the second order with respect to $\varkappa$.

The results of this section show that, in the cases of such small attenuations, for which it is still reasonable to introduce the radiation

1 Let us note that to the function (8.27) corresponds the following function $U(p)$, if we use the expansion into Fourier integrals:

$$U(p) = \frac{3C}{p^3 a^3}(\sin pa - pa \cos pa).$$

ability and intensity inside the absorbing medium (i.e., with a precision up to $\varkappa^2$, section 5), the equilibrium intensity in the medium is expressible to the same degree of precision by the formula for transparent media. The same refers to the densities of electric and magnetic energies of the radiation field. In some of the cases, the presence of terms of the order $\varkappa$ and independent of a in I_ω does not contradict the conclusion arrived at in view of the incompletely defined separation of energy quantities into parts, corresponding to radiation and internal energy of the medium.

Section 9. Magnetic Losses and Fluctuations of the Lateral Magnetic Field

In sections 6 and 7 we convinced ourselves that the assumption (5) concerning the form of the correlation functions for the components of the lateral electric field $\vec{K}$, namely

$$F_{\alpha\beta}(r) = \overline{K_\alpha(\vec{r}')K^*_\beta(\vec{r}'')} = C\delta_{\alpha\beta}\,\delta(\vec{r}), \qquad (r = |\vec{r}' - \vec{r}''|), \tag{9.1}$$

permits, with a corresponding choice of the constant C, to describe correctly the radiation of an absorbing medium in the interior of space, and for equilibrium radiation in a transparent medium with $n^2 = \varepsilon_1\mu_1 > 0$ it gives the law (6.18). As a radiation source we studied an attenuating medium, having only electric losses, and took the magnetic permittivity μ to be real. If we throw off this restriction and carry out the evaluation, assuming ε as well as μ to be complex, but introducing as before only the electric lateral field $\vec{K}$, then we obtain for the radiation intensity in vacuum an expression which can no longer be reduced to the form (6.14). The possibility of representing the intensity in the form governed by Kirchhoff's law can again only be achieved by introducing, along with $\vec{K}$, the lateral magnetic fluctuating field $\vec{M}$.

Keeping in mind the symmetric form of the field equation (1.9)

$$\begin{aligned} \operatorname{curl} \vec{E} &= -ik\mu\vec{H} - ik(\mu-1)\vec{M} \\ \operatorname{curl} \vec{H} &= ik\varepsilon E + ik(\varepsilon-1)\vec{K}\,, \end{aligned} \tag{9.2}$$

it is natural to make the following assumptions concerning the statistical properties of $\vec{M}$:

1) the correlation function for the components of $\vec{M}$ also has the form (9.1), i.e.,

$$\overline{M_\alpha(\vec{r}') M^*_\beta(\vec{r}'')} = D\delta_{\alpha\beta}\delta(\vec{r}) ; \qquad (9.3)$$

2) the lateral random fields $\vec{K}$ and $\vec{M}$ are not correlated between themselves

$$\overline{K_\alpha(\vec{r}') M^*_\beta(\vec{r}'')} = 0 . \qquad (9.4)$$

With these assumptions, the scheme for calculating the radiation intensity in vacuum due to the absorbing medium in the half-space $z < 0$, remains the same as in Appendix II and requires only a few very obvious supplements. Therefore we shall not give the calculations but shall immediately give its result:

$$I_\omega = \frac{k^3 ci \cos\theta}{8\pi^3(\zeta^* - \zeta)} \left\{ C\,|\varepsilon - 1|^2 \left(\frac{|\mu|^2}{|\mu\cos\theta + \zeta|^2} + \frac{\sin^2\theta + |\zeta|^2}{|\varepsilon\cos\theta + \zeta|^2} \right) + \right.$$

$$\left. + D\,|\mu - 1|^2 \left(\frac{\sin^2\theta + |\zeta|^2}{|\mu\cos\theta + \zeta|^2} + \frac{|\varepsilon|^2}{|\varepsilon\cos\theta + \zeta|^2} \right) \right\} .$$

For brevity, the following notation is used here

$$\zeta = \sqrt{\varepsilon\mu - \sin^2\theta} .$$

With real μ and $D = 0$ the written expression for I_ω goes over into formula (1.15) of Appendix I.

But for complex ε and μ.

$$1 - R_{\perp} = 1 - \left|\frac{\mu\cos\theta - \zeta}{\mu\cos\theta + \zeta}\right|^2 = \frac{2\cos\theta(\mu^*\zeta + \mu\zeta^*)}{|\mu\cos\theta + \zeta|^2},$$

$$1 - R_{\parallel} = 1 - \left|\frac{\varepsilon\cos\theta - \zeta}{\varepsilon\cos\theta + \zeta}\right|^2 = \frac{2\cos\theta(\varepsilon^*\zeta + \varepsilon\zeta^*)}{|\varepsilon\cos\theta + \zeta|^2},$$

whence

$$\frac{\cos\theta}{|\mu\cos\theta + \zeta|^2} = \frac{1 - R_{\perp}}{2(\mu\zeta^* + \mu^*\zeta)}, \qquad \frac{\cos\theta}{|\varepsilon\cos\theta + \zeta|^2} = \frac{1 - R_{\parallel}}{2(\varepsilon\zeta^* + \varepsilon^*\zeta)}$$

Substituting this into I_{ω} and considering that

$$(\zeta^* - \zeta)(\mu^*\zeta + \mu\zeta^*) = (\mu^* - \mu)(\sin^2\theta + |\zeta|^2) + (\varepsilon^* - \varepsilon)|\mu|^2,$$

$$(\zeta^* - \zeta)(\varepsilon^*\zeta + \varepsilon\zeta^*) = (\varepsilon^* - \varepsilon)(\sin^2\theta + |\zeta|^2) + (\mu^* - \mu)|\varepsilon|^2,$$

we obtain

$$\ldots \frac{k^3 c_1}{\ldots \pi}\left\{\frac{1 - R_{\perp}}{2} \cdot \frac{|\mu|^2 C|\varepsilon - 1|^2 + (\sin^2\theta + |\zeta|^2)D|\mu - 1|^2}{|\mu|^2(\varepsilon^* - \varepsilon) + (\sin^2\theta + |\zeta|^2)(\mu^* - \mu)}\right.$$

$$\left. + \frac{1 - R_{\parallel}}{2} \cdot \frac{(\sin^2\theta + |\zeta|^2)C|\varepsilon - 1|^2 + |\varepsilon|^2 D|\mu - 1|^2}{(\sin^2\theta + |\zeta|^2)(\varepsilon^* - \varepsilon) + |\varepsilon|^2(\mu^* - \mu)}\right\}.$$

... this expression assume the form (6.14)

$$\ldots(1 - R) = I_{0\omega}\left(\frac{1 - R_{\perp}}{2} + \frac{1 - R_{\parallel}}{2}\right)$$

$$C|\varepsilon - 1|^2 = A(\varepsilon^* - \varepsilon),$$

$$D|\mu - 1|^2 = A(\mu^* - \mu),$$

$$\frac{k^3 ci}{8\pi^3} A = I_{0\omega} \cdot$$

whence

$$C = \frac{16\pi^3}{k^3 c} I_{0\omega} \mathrm{Im}\left(\frac{1}{\varepsilon - 1}\right), \quad D = \frac{16\pi^3}{k^3 c} I_0 \mathrm{Im}\left(\frac{1}{\mu - 1}\right). \quad (9.5)$$

Thus, the presence of magnetic losses absolutely requires the introduction of the fluctuating magnetic lateral field $\vec{M}$, uncorrelatable with the electric lateral field $\vec{K}$ and having the same δ-correlation components as $\vec{K}$. The complex-ness of μ does not influence the value of the constant C of the electric lateral field and conversely -- the complex-ness of ε has no effect on D. With real ε (or μ) we have $C = 0$ (correspondingly, $D = 0$).

These conclusions apply to the case where the field equations are taken in the symmetric form (9.2). As was pointed out in section 1, with an appropriate change in the determination of the intensity of the microfield, equations (9.2) can be replaced by others, into which the fluctuation "force" enters in a different manner. Thus, for instance, in the equations (1.8), the "force" is contained only in the second equation

$$\mathrm{curl}\, \vec{E} = -ik\vec{H}$$

$$\mathrm{curl}\, \frac{\vec{H}}{\mu} = ik\varepsilon\vec{E} + \vec{P} \quad (9.6)$$

where

$$\vec{P} = ik(\varepsilon - 1)\vec{K} + \mathrm{curl}\, \frac{\mu - 1}{\mu}\vec{M}.$$

From this expression for $\vec{P}$ it is clear that the simplification of the equation is related to the complication of the form of the correlation

function of lateral "forces" In fact, taking (9.1), (9.3) and (9.4) into account, it is easy to obtain

$$\overline{P_{\alpha}(\vec{r}')P^{*}_{\beta}(\vec{r}'')} = \left\{ k^2 |\varepsilon - 1|^2 C\delta(\vec{r}) - D\nabla^2 \left[\left| \frac{\mu - 1}{\mu} \right|^2 \delta(\vec{r}) \right] \right\} \delta_{\alpha\beta} +$$

$$+ D \frac{\partial^2}{\partial x_{\alpha} \partial x_{\beta}} \left[\left| \frac{\mu - 1}{\mu} \right|^2 \delta(\vec{r}) \right] .$$

Consequently, in equation (9.6) the complexness μ will entrain the necessary application of the general expression (3.3) for the correlation function component $\vec{P}$. Due to this complication of calculations, it becomes necessary in many cases to prefer the symmetric form of the field equation (9.2).

In all that is to follow we shall again assume that magnetic losses are absent and accordingly shall introduce only the electric lateral field.

Section 10. Radiation of a Cylinder

We had studied the thermal radiation of an absorbing medium, occupying an infinite half-space, either into an unbounded adjacent half-space (section 6), or into a wave guide limited by a plane mirror parallel to the surface of the medium (section 7). Both problems contain all the elements of importance for the theory under study, namely the solution of the border problem with non-uniform field equations in the interior of the absorbing medium and uniform equations for the external space with the consequent over-all averaging of energy quantities by means of the space correlation function of the lateral fluctuating field. The solution of the border problem, of course, embraces all diffraction phenomena which occur under the given geometric conditions and which acquire prime importance for those cases in which the body dimensions are comparable to the wave length.

In this respect the examinations of the problem are not typical at all. With the aim of clarifying a series of basic problems, we handled

rather simple geometric conditions, assuming that the radiating medium occupies an infinite half-space. However, since the dimensions or distances (width of the wave guide or simply the distance between the point of interest and the surface of the radiating medium) which we considered were made comparable to the wave length, we obtained results which do not lie within the field of vision of the classical theory of radiation [quasi-stationary thermal field (section 6), interference deviation in equilibrium radiation from homogeneity and isotropicity (section 7)].

However, the application of theory is of greater interest to those problems in which the dimensions of the radiating body are not very large compared to the wave length. In this section we shall study one of the problems belonging precisely to this class: thermal radiation of a circular cylinder, infinitely long, but with an arbitrary ratio between the radius and the wave length.

The primary field $\vec{E}_0$, $\vec{H}_0$ in the interior of the cylinder satisfies equations (1.10)

$$\text{curl } \vec{E}_0 = -ik\mu\vec{H}_0 \tag{10.1}$$

$$\text{curl } \vec{H}_0 = ik\varepsilon\vec{E}_0 + ik(\varepsilon - 1)\vec{K}.$$

To obtain the general solution one must add to the particular solution $\vec{E}_0$, $\vec{H}_0$ of these non-uniform equations the general solution $\vec{E}_1$, $\vec{H}_1$ of the uniform equations ($\vec{K} = 0$), satisfying the conditions of regularity along the cylinder axis.[1] For the external space (for simplicity -- vacuum) we have the equations

$$\text{curl } \vec{E} = -ik\vec{H} \tag{10.2}$$

$$\text{curl } \vec{H} = ik\vec{E}$$

whose solution must satisfy the radiation condition, i.e., must consist of waves propagating from the cylinder.

1. In the case of a semi-infinite medium, we called this field the reflected field and denoted it by $\vec{E}_r$, $\vec{H}_r$ (section 6).

In cylinder coordinates r, φ, z we have the following three fundamental vector functions, by means of which the solution of equation (10.1) is expressible.[1]

$$\bar{M}_n(h) = e^{i(n\varphi + hz)}\left(\frac{inZ}{r}\bar{i}_1 - Z'\bar{i}_2\right),$$

$$\bar{N}_n(h) = e^{i(n\varphi + hz)}\left(\frac{ihZ'}{q}\bar{i}_1 - \frac{hnZ}{qr}\bar{i}_2 + \frac{\lambda^2 Z}{q}\bar{i}_3\right), \qquad (10.3)$$

$$\bar{L}_n(h) = e^{i(n\varphi + hz)}\left(Z'\bar{i}_1 + \frac{inZ}{r}\bar{i}_2 + ih\,Z\,\bar{i}_3\right),$$

where $Z = Z_{|n|}(\lambda r)$ is any cylinder function of the order $|n|$,

$$\lambda = \sqrt{q^2 - h^2}, \qquad q = k\sqrt{\epsilon\mu} \qquad (10.4)$$

and $\bar{i}_1, \bar{i}_2, \bar{i}_3$ are the unit vectors with respect to the coordinates r, φ and z. Primes indicate differentiation with respect to r. The functions (10.3) are such that

$$\begin{aligned} \text{curl}\ \bar{M} &= q\,\bar{N} \\ \text{curl}\ \bar{N} &= q\,\bar{M} \qquad (10.5) \\ \text{curl}\ \bar{L} &= 0 \end{aligned}$$

If div $\bar{E} = 0$ (case of uniform equations), then $\bar{E}$ must be expressed only in terms of the functions $\bar{M}$ and $\bar{N}$. For non-uniform equations, however, when div $\bar{E} \neq 0$, the solution must contain all three fundamental functions and will thus be expressed in the form of the following Fourier series

1. See J. A. Stratton. Theory of Electromagnetism, section 7.2. Stratton introduces even and uneven functions containing cos $n\varphi$, and sin $n\varphi$ and indicates that the introduction of complex relations ($e^{in\varphi}$) leads to difficulties. In fact, there are here no difficulties at all, and on the contrary all formulae become half as short.

in φ and Fourier integral in z:

$$\vec{E}_0 = \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dh\,(A\vec{M} + B\vec{N} + C\vec{L} + \tilde{A}\tilde{\vec{M}} + \tilde{B}\tilde{\vec{N}} + \tilde{C}\tilde{\vec{L}}), \tag{10.6}$$

$$\vec{H}_0 = \gamma \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dh\,(A\vec{N} + B\vec{M} + \tilde{A}\tilde{\vec{N}} + \tilde{B}\tilde{\vec{M}}),$$

where

$$\gamma = i\sqrt{\frac{\varepsilon}{\mu}} \tag{10.7}$$

By $\vec{M}$, $\vec{N}$ and $\vec{L}$ are understood functions in which $Z = J_{|n|}(\lambda r)$, and the curl denotes functions in which $Z = N_{|n|}(\lambda r)$. In this, the relation of A, B, ..., $\tilde{C}$ to r must be found by the method of variation of these constants (for brevity, the index n and the argument h are omitted in both the fundamental functions and the "constants" A, B, ..., $\tilde{C}$).

In accordance with the expansion (10.6), we analogously represent the lateral field $\vec{K}$, assuming

$$\frac{\varepsilon - 1}{\varepsilon}\vec{K} = \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dh\, e^{i(n\varphi + hz)}\, \vec{G}_n(h, r), \tag{10.8}$$

whence

$$\vec{G}_n(h, r) = \frac{\varepsilon - 1}{4\pi^2\varepsilon} \int_{-\infty}^{+\infty} dz \int_{-\pi}^{+\pi} d\varphi\, e^{-i(n\varphi + hz)}\, \vec{K}. \tag{10.9}$$

The general solution $\vec{E}_1$, $\vec{H}_1$ of the uniform equations (10.1), since for it div $\vec{E}_1 = 0$, contains by virtue of the regularity condition along r = 0 only the functions $\vec{M}$ and $\vec{N}$, in which $Z = J_{|n|}(\lambda r)$:

$$\vec{E}_1 = \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dh\,(A_1\vec{M} + B_1\vec{N}), \tag{10.10}$$

$$\vec{H}_1 = \gamma \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dh\,(A_1\vec{N} + B_1\vec{M}) . \tag{10.10}$$

Finally we have in the external space the solution of equation (10.2)

$$\vec{E} = \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dh(P\,\vec{M}^e + Q\,\vec{N}^e) ,$$

$$\vec{H} = i \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dh(P\,\vec{N}^e + Q\vec{M}^e), \tag{10.11}$$

where $\vec{M}^e$ and $\vec{N}^e$ denote the functions (10.3) containing, in accordance with the radiation condition, $Z = H \equiv H_n^{(2)}(\lambda_0 r)$, where $q = k$ and

$$\lambda_0 = \sqrt{k^2 - h^2} \tag{10.12}$$

On the cylinder surface ($r = a$), the boundary conditions (1.13) must be satisfied, taking in one case the form

$$\begin{aligned} &E_{0\varphi} + E_{1\varphi} = E_{\varphi}, \quad H_{0\varphi} + H_{1\varphi} = H_{\varphi}, \\ &E_{0z} + E_{1z} = E_z , \quad H_{0z} + H_{1z} = H_z \end{aligned} \quad (r = a). \tag{10.13}$$

These conditions permit expressing the constants P and Q in terms of the values of A and B on the cylinder surface; the latter, in their turn, can be expressed in a definite way in terms of the components of the vector $\vec{G}$ for which we have formula (10.9). The correlation function for K

$$\overline{K_{\alpha}(r, \varphi, z)K^*_{\beta}(r_1, \varphi_1, z_1)} = C\,\delta_{\alpha\beta}\,\delta(r - r_1)\,\frac{\delta(\varphi - \varphi_1)}{r}\,\delta(z - z_1),$$

$$(\alpha, \beta = r, \varphi, z) \tag{10.14}$$

determines, of course, the correlation function for $\vec{G}$. With the help of

(10.9) and (10.14) it is not difficult to obtain

$$\overline{G_{n\alpha}(h, r)G^*_{m\beta}(h_1, r_1)} = \frac{|\varepsilon - 1|^2 c}{4\pi^2|\varepsilon|^2}\mathcal{E}_{nm}\,\delta_{\alpha\beta}\,\frac{\delta(r - r_1)}{r}\,\delta(h - h_1),$$

$$(\alpha, \beta = r, \varphi, z). \tag{10.15}$$

In sum, it is possible for us to evaluate the average values $\overline{|P|^2}$ and $\overline{|Q|^2}$, by means of which the radiation power per unit cylinder length is expressed,

$$P_\omega = \frac{4c}{k}\sum_{n=-\infty}^{+\infty}\int_{-k}^{+k} dh \int_{-\infty}^{+\infty} dh_1 \lambda_0^2\,(\overline{|P|^2} + \overline{|Q|^2})\,. \tag{10.16}$$

All the evaluations indicated here [including the derivation of (10.16)] are carried out in detail in Appendix IV and give the following result for the power P_ω :

$$P_\omega = \frac{4\mu(\varepsilon^* - \varepsilon)I_{0\omega}}{ik^2a^2}\sum_{n=-\infty}^{+\infty}\int_{-k}^{+k}\frac{\lambda_0^2\,dh}{|\lambda|^4|\Delta|^2}\,\times$$

$$\times\left\{\frac{a}{\lambda^2 - \lambda^{*2}}\left[k^2\mu(|\Delta_2|^2 + |\delta|^2)(\lambda^2JJ'^* - \lambda^{*2}J^*J') + \right.\right.$$

$$\left. + \frac{|\Delta_1|^2 + |\delta|^2}{\mu}\left(h^2(\lambda^2JJ'^* - \lambda^{*2}J^*J') + |\lambda|^4(JJ'^* - J^*J')\right)\right] -$$

$$\left. - khnJJ^*\left[(\Delta_2^* + \Delta_1^*)\delta + (\Delta_2 + \Delta_1)\delta^*\right]\right\} \tag{10.17}$$

Here

$$\Delta = \Delta_1\Delta_2 - \delta^2, \qquad \Delta_1 = H'J - \frac{iq\lambda_0^2}{\gamma k\lambda^2}HJ',$$

$$\Delta_2 = H'J - \frac{\gamma q \lambda_0^2}{ik\lambda^2} HJ',$$

$$\delta = \frac{hn}{ka}\left(1 - \frac{\lambda_0^2}{\lambda^2}\right) HJ, \qquad J = J_{|n|}(\lambda a), \qquad H = H^{(2)}_{|n|}(\lambda_0 a), \tag{10.18}$$

$$\lambda^2 = q^2 - h^2, \qquad \lambda_0^2 = k^2 - h^2, \qquad q = k\sqrt{\varepsilon\mu}$$

and the prime indicates differentiation with respect to a.

Formula (10.17) can be written in a more compact and physically better understandable form.

To fixed values of n and h correspond two definite types of partial waves, radiated by the cylinder into external space. These waves are expressed by means of functions with coefficients P and Q in the expressions in (10.11) under the integral sign and it is convenient to call them P- and Q-waves. If we replace in $\vec{H}^e$ and $\vec{N}^e$ the Haenckel function of the second kind by the same function of the first kind, we obtain P- and Q-waves falling on the cylinder, or, in better words, contracting towards it. The P-waves, for which at least one of the arguments n or h is different, are orthogonal between themselves. The same applies to the Q-waves, and between P- and Q-waves orthogonality exists for different as well as for coinciding n and h. Any wave field, which contracts toward the cylinder, can be separated into the conjugates of travelling waves of the type P and Q with values of n from $-\infty$ to $+\infty$ and h from -k to +k.

For a high conductivity of the cylinder, the incident P- (or Q-) wave gives also in the reflected field only a P- (correspondingly Q-) wave, but in the general case, the incidence of waves of the P- or Q-type causes a mixing of P- and Q-waves with the same n and h in the reflected field. This, of course, does not hinder the determination of the energy coefficients of attenuation for incident P- or Q-waves (by difference of the energy flows to the cylinder and from it). With the help of these absorption coefficients $A_{Pn}(h)$ and $A_{Qn}(h)$, evaluated in Appendix IV,

(10.17) can be written in the form[1]

$$P_\omega = \frac{2\pi^2 I_{0\omega}}{k^2} \sum_{n=-\infty}^{+\infty} \int_{-k}^{+k} \left\{ A_{Pn}(h) + A_{Qn}(h) \right\} \frac{dh}{2\pi} . \qquad (10.19)$$

In the region of validity of the Rayleigh-Jeans law when, according to (5.12), $I_{0\omega} = \frac{\theta k^2}{4\pi^3}$, we obtain

$$P_\omega = \frac{\theta}{2\pi} \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left\{ A_{Pn}(h) + A_{Qn}(h) \right\} \frac{dh}{2\pi} . \qquad (10.20)$$

The derivation of formula (10.19), despite the fact that it has been carried out with the help of a special form of the function, characteristic of the given concrete problem, permits the understanding in (10.19) of a particular expression for the following general situation. If the wave field incident on the body can be expanded onto the conjugate of mutually non-interfering (mutually orthogonal) travelling waves and if A_i is the energy absorption coefficient of the given body for a single wave from this conjugate, then the thermal emissive power of the body at frequency ω is

$$P_\omega = \frac{2\pi^2 I_{0\omega}}{k^2} \sum_i A_i , \qquad (10.21)$$

where the sum extends to all running waves of the conjugate.

We shall not prove this (physically quite clear) situation in the

1. The multiplier $1/2\pi$ has been left under the integral sign for the following reasons. If we were to use the expansion with respect to z not into a Fourier integral, but into a series with some period ℓ, then we would have a discrete value for h: $h_m = 2\pi m/\ell$. Therefore, $\Delta m = 1 = \frac{\ell \cdot \Delta h}{2\pi}$, i.e., the integral with the multiplier $1/2\pi$ for discrete values of h is transformed into a sum of absorption coefficients for these values of $h = h_m$.

most general case. We can convince ourselves later of its validity when solving other problems (sections 14 and 15), and besides, with respect to the propagation of thermal radiation in lines and wave guides, we shall give its general proof, based on classical theory (section 17).

It should be noted that precisely for wave guides the individual terms in (10.21) can be, up to certain limits, separated out and thereby acquire a direct physical meaning. In the problem studied concerning radiation of a cylinder the separation of partial waves is practically not realizable and the whole sum taken together is of interest. This sum, besides depending on the electric parameters of the substance of the cylinder, depends on the ratio a/λ, which is of a great, new moment compared to the result which could be obtained on the basis of the usual Kirchhoff laws and which is limited by the condition $a \gg \lambda$.

The approximate estimates carried out in Appendix IV give the following value for the thermal emissive power per unit area of cylinder surface, i.e., for $P_\omega = \frac{\bar{P}_\omega}{2\pi a}$ [we take here the region of applicability of the Rayleigh-Jeans law: $I_{0\omega} = \frac{\theta k^2}{4\pi^3}$]. If λ is the wave-length in vacuum, and $d = c/\sqrt{2\pi\sigma\mu\omega}$ is the thickness of skin-layer, then for $a \gg \lambda \gg d$ (thick, well conducting cylinder)

$$P_\omega = \frac{2\theta}{3\pi^2} k^3 d\mu ; \qquad (10.22)$$

for $\lambda \gg a \gg d$ (thick, well conducting cylinder)

$$P_\omega = \frac{\theta k}{4\pi^2 a} \sqrt{\frac{\mu d}{a|\ln(ka/2)|^3}} \qquad (10.23)$$

and for $\lambda \gg a|\sqrt{\varepsilon\mu}|$ (thin, poorly conducting cylinder)

$$P_\omega = \frac{\theta}{6\pi^2} k^3 a \varepsilon'' . \qquad (10.24)$$

Formula (10.22) is the power, emitted by unit area of any well conducting surface for the case of such short waves that approximations of geometric optics are valid. Formula (10.22) is a direct consequence of the usual Kirchhoff law. The other two formulae refer to the opposite case ($\lambda \gg a$) and give the relation of unit power to the cylinder radius a. Formula (10.23) embodies the transition to the ideal conductor ($\sigma \to \infty$, $d \to 0$), formula (10.24) the transition to the ideal dielectric ($\varepsilon'' \to 0$).

CHAPTER III. SURFACE RANDOM EMF

Section 11. Boundary Conditions of M. A. Leontovich. Introduction of Surface Lateral Field

The problems of radiation of half-space and of the cylinder, examined in the previous chapter, give a sufficiently good picture of how the determination of the external electro-magnetic field is complicated by the necessity of studying the field in the interior of a radiating body.

In the cases, when it is necessary to find the field exterior to bodies at which the skin-effect for a given frequency is sufficiently strongly developed, the general exposition of electro-dynamic problems can be, as is known, appreciably simplified. Instead of finding the solution with the help of boundary conditions (1.13), the approximate boundary conditions for fields exterior to the bodies can be utilized, and, thus, one can limit oneself to the study of this external field. Approximate relations, whose possible utilization for boundary conditions were pointed out by M. A. Leontovich[1] in 1940, have been successfully applied in a series of works.[2] These conditions can be obtained either by means of the solution of the problem of skin-effect by the following method of perturbation[3] or on the basis of simple visual considerations (see below). The first method, of course, also gives an estimate of the precision of these asymptotic boundary conditions, valid for

1. See article by M. A. Leontovich in Collection II "Investigation of radio wave propagation", p. 5 (M., 1948). Analogous relations were simultaneously deduced by A. N. Shukin ["Propagation of radio waves", Part I, p. 52 (M., 1940)] and were utilized for a series of calculations, but not as general boundary conditions for a specific type of boundary problems.

2. Ia. L. Alpert, JTF 10, 1358, 1940; G. A. Greenberg, J. of Phys. (USSR) 6, 185, 1942; M. A. Leontovich, Izv. AN USSR (Ser. Phys.) 8, 16, 1944; M. E. Jabotinski, M. L. Levin and S. M. Rytov, JTF 20, 257, 1950.

3. S. M. Rytov, JETF 10, 180, 1940.

$$kd \ll 1 ,$$

where $k = \frac{\omega}{c} = \frac{2\pi}{\lambda}$, and d is the thickness of the plane skin-layer

$$d = \frac{c}{\sqrt{2\pi\sigma\mu\omega}} \qquad (11.1)$$

(σ is the conductivity). If the surface is curved, then the conditions are valid with a precision up to the first order with respect to kd, and for a plane surface, up to the second order inclusive. Understandably, in the first case the precision is the higher, the greater the radius of curvature of the surface compared to d.

The simple considerations, leading to the Leontovich boundary conditions, are reduced to the application of Ohm's law and of the boundary condition for the magnetic field in the presence of a strong skin-effect.

Let $\vec{j}$ and $\vec{i}$ denote, respectively, the volume and surface current density. We have for the tangential component of the magnetic field on the exterior side of the surface of the body

$$\vec{H}_t = -\frac{4\pi}{c}[\vec{N}, \vec{i}] , \qquad (11.2)$$

where $\vec{N}$ is the external normal to the surface. Let us assume, for simplicity, that the surface is plane and let the z-axis be in the direction of $\vec{N}$. For a strong skin-effect, we can let

$$\vec{j} = \vec{j}_t(x, y)e^{ik\sqrt{\varepsilon\mu}\, z} , \qquad (11.3)$$

where $\vec{j}_t(x, y)$ is the value of $\vec{j}$ for $z = 0$ (on the surface). Therefore,

$$\vec{i} = \int_{-\infty}^{0} \vec{j}\, dz = \frac{\vec{j}_t}{ik\sqrt{\varepsilon\mu}} . \qquad (11.4)$$

But, according to Ohm's law

$$\vec{j} = \frac{(\varepsilon - 1)i\omega}{4\pi}\vec{E} \approx \frac{i\omega\varepsilon}{4\pi}\vec{E}$$

where we neglected unity since, according to assumption, $|\varepsilon| >> 1$. Thus,

$$\vec{j}_t = \frac{i\omega\varepsilon}{4\pi}\vec{E}_t , \qquad (11.5)$$

where $\vec{E}_t$ is the value of the tangential componant of the electric field on the inner side of the body surface, and -- in view of the continuity of tangential components -- is equal to the surface value of the tangential component of the external electric field. Substituting (11.4) and (11.5) in (11.2), we obtain, according to Leontovich's condition

$$\sqrt{\mu}\vec{H}_t = -\sqrt{\varepsilon}[\vec{N}, \vec{E}_t] \qquad (z = 0). \qquad (11.6)$$

As shown in (11.6), connecting the values on the surface of the body with the tangential component of the external field, it is permissible to go on without working out the problem of the field inside of the bodies. The result thus obtained for the external field will be the more precise, the stronger the skin-effect and, consequently, the closer the structure of the external field to that which it would have had in the case of ideally conducting bodies of the same shape, dimensions and position as the body under study.

Evidently, it would have been very logical to utilize the properties, given by conditions (11.6), as well as in the questions of thermal radiation of interest to us. The bases for this are particularly frequent in questions related to "radio noises" where in many cases the radiation sources are bodies with metallic conductivity and the frequencies are so high that the skin-effect is very strongly developed. It is natural to assume, that the necessary generalization of condition (11.6) must consist of the introduction of the surface lateral random field $\mathcal{K}$, acting in a sum with $\vec{E}_t$

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what we had clarified in section 9, beside the electric lateral field one must also introduce the as yet uncorrelated to it magnetic lateral field. Let us denote the potential of this surface field by $\mathcal{M}$. In lieu of (11.7) we must then write

$$\sqrt{\mu}\,(\vec{H}_t + \vec{\mathcal{M}}) = -\sqrt{\varepsilon}\,[\vec{N}, \vec{E}_t + \vec{\mathcal{K}}], \qquad (z = 0).$$

In the case of small values of $|\varepsilon|$ and for $|\mu| \gg 1$ this gives the condition, symmetrical to (11.8),

$$\vec{H}_t = -\vec{\mathcal{M}} \qquad (z = 0).$$

However in what is to follow we shall as before assume that μ is real and correspondingly shall not introduce $\vec{\mathcal{M}}$.

We must now establish the form of the correlation function for the component $\vec{\mathcal{K}}$.

Section 12. Correlation Function for the Surface Field

The establishment of the correlation function component of $\vec{\mathcal{K}}$ or, if this field is presented in the form of a double Fourier integral

$$\mathcal{K} = \int_{-\infty}^{+\infty} \varrho(p_1, p_2)\, e^{i(p_1x+p_2y)}\, dp_1\, dp_2, \qquad (12.1)$$

of the correlation function component conjugate to Fourier ϱ does not require any new physical assumptions. The relations (11.7) [and for very large $|\varepsilon|$ -- (11.8)] determine $\vec{\mathcal{K}}$ in terms of the full potentials external to the medium, which, in turn, are expressed by the volume lateral field $\vec{K}$. Thereby, for the determination of the correlation function component $\mathcal{K}$

$$\mathcal{F}_{\alpha\beta} = \overline{\mathcal{K}_\alpha(x', y')\,\mathcal{K}^*_\beta(x'', y'')}, \qquad (\alpha, \beta = 1,2) \qquad (12.2)$$

or the component ϱ

$$\mathcal{J}_{\alpha\beta} = \overline{g_{\alpha}(p_1', p_2')g^*_{\beta}(p_1'', p_2'')}, \quad (\alpha, \beta = 1,2) \qquad (12.3)$$

we can use the solution of any problem, in which the field external to the medium has been found in terms of $\vec{K}$ (or $\vec{g}$). In particular, we can take the solution studied in section 6 of the problem concerning radiation of the medium, occupying a half-space, into vacuum. We shall utilize now the results obtained there.

For the field in vacuum we had the expansion according to plane waves

$$\begin{aligned} \vec{E} &= \int_{-\infty}^{+\infty} \vec{v}(p_1, p_2) e^{i\vec{t}\cdot\vec{r}}\, dp_1\, dp_2 \\ \vec{H} &= -\frac{1}{k}\int_{-\infty}^{+\infty} [\vec{t}, \vec{v}]\, e^{i\vec{t}\cdot\vec{r}}\, dp_1\, dp_2 \end{aligned} \qquad (12.4)$$

where

$$t_1 = p_1, \quad t_2 = p_2, \quad t_3 = -\sqrt{k^2 - (p_1^2 + p_2^2)}$$

and $\vec{v}(p_1, p_2)$ is expressed in terms of $\vec{g}$ by formulas (I.5) obtained in Appendix I

$$\left.\begin{matrix} v_1 \\ v_2 \\ v_3 \end{matrix}\right\} = -\frac{k^2\mu(\varepsilon - 1)s_3}{\Delta}\int_{-\infty}^{+\infty}\frac{dp_3}{p_3^2 - s_3^2}\, \times$$

$$\times \begin{cases} \frac{1}{\alpha}[(p_2^2\beta - s_3t_3\alpha)g_1 - p_1p_2\beta g_2 - p_1t_3\alpha g_3], \\ \frac{1}{\alpha}[-p_1p_2\beta g_1 + (p_1^2\beta - s_3t_3\alpha)g_2 - p_2t_3\alpha g_3], \\ [s_3(p_1g_1 + p_2g_2) + (p_1^2 + p_2^2)g_3]. \end{cases} \qquad (12.5)$$

Here

$$\alpha = \mu t_3 - s_3, \qquad \beta = t_3 - \mu s_3,$$

$$\Delta = (p_1^2 + p_2^2)\beta - s_3 t_3 \alpha, \tag{12.6}$$

$$s_3 = +\sqrt{k^2 \varepsilon \mu - (p_1^2 + p_2^2)} .$$

We shall examine only the first term of the asymptotic (as $|\varepsilon| \to \infty$) expansion of $\mathcal{K}$. In this approximation we can restrict ourselves to the limiting form of the boundary conditions (11.8)

$$\mathcal{K}_\alpha = - E_\alpha , \quad (\alpha = 1,2). \tag{12.7}$$

With this degree of precision for any finite p_1 and p_2 we have

$$s_3 \approx k\sqrt{\varepsilon \mu} ,$$

so that in (12.6)

$$\alpha \approx - s_3, \qquad \beta \approx - \mu s_3, \qquad \Delta \approx s_3^2 t_3.$$

Retaining in formulae (12.5) only the terms senior with respect to s_3, we obtain expressions for v_1 and v_2

$$v_\alpha = k^2 \varepsilon \mu \int_{-\infty}^{+\infty} \frac{E_\alpha \, dp_3}{p_3^2 - s_3^2} , \quad (\alpha = 1,2). \tag{12.8}$$

Finally, from (12.4), (12.7) and (12.8) we find

$$\mathcal{K}_\alpha = - E_\alpha = - k^2 \varepsilon \mu \int_{-\infty}^{+\infty} e^{i(p_1 x + p_2 y)} \, dp_1 dp_2 \int_{-\infty}^{+\infty} \frac{E_\alpha \, dp_3}{p_3^2 - s_3^2} \quad (\alpha = 1,2), \tag{12.9}$$

whence it follows [see (12.1)], that

$$g_\alpha = -k^2\varepsilon\mu \int_{-\infty}^{+\infty} \frac{g_\alpha \, dp_3}{p_3^2 - s_3^2}, \quad (\alpha = 1,2). \tag{12.10}$$

Using the correlation function (4.10) for component $\vec{g}$

$$\overline{g_\alpha(\vec{p}\,')g_\beta^*(\vec{p}\,'')} = \frac{2C\delta_{\alpha\beta}}{(2\pi)^3}\delta(\vec{p}\,' - \vec{p}\,''),$$

we obtain

$$\mathcal{J}_{\alpha\beta} = \overline{g_\alpha g_\beta^*} = k^4|\varepsilon|^2\mu^2 \iint_{-\infty}^{+\infty} \frac{\overline{g_\alpha g_\beta^*}\, dp_3'\, dp_3''}{(p_3'^2 - s_3^2)(p_3''^2 - s_3^{*2})} =$$

$$= \frac{2C\delta_{\alpha\beta}\, k^4|\varepsilon|^2\mu^2}{(2\pi)^3}\delta(p_1' - p_1'')\,\delta(p_2' - p_2'') \int_{-\infty}^{+\infty} \frac{dp_3'}{|p_3'^2 - s_3^2|^2}.$$

The value of the integral used above and found in Appendix I is

$$\int_{-\infty}^{+\infty} \frac{dp_3'}{|p_3'^2 - s_3^2|^2} = \frac{i\pi}{|s_3|^2(s_3^* - s_3)} \approx \frac{i\pi}{k^3|\varepsilon|\mu(\sqrt{\varepsilon^*\mu} - \sqrt{\varepsilon\mu})},$$

so that finally

$$\mathcal{J}_{\alpha\beta} = \frac{\mathcal{C}\delta_{\alpha\beta}}{(2\pi)^2}\delta(p_1' - p_1'')\,\delta(p_2' - p_2''), \tag{12.11}$$

where

$$\mathcal{C} = \frac{iC}{\dfrac{1}{k\sqrt{\varepsilon\mu}} - \dfrac{1}{k\sqrt{\varepsilon^*\mu}}}. \tag{12.12}$$

From (12.1) and (12.11) it immediately follows that the correlation function component of $\mathcal{K}$ is

$$\mathcal{F}_{\alpha\beta} = \mathcal{C}\,\delta_{\alpha\beta}\,\delta(x' - x'')\,\delta(y' - y''). \tag{12.13}$$

The ratio (12.12) between the correlation function constants of the surface and of the volume can be given a very simple form if we note that

$$\frac{1}{k\sqrt{\varepsilon\mu}} = \frac{d}{1 - i}, \tag{12.14}$$

where d is the thickness of the plane skin-layer. By virtue of (12.14) formula (12.12) gives

$$\mathcal{C} = \frac{C}{d}. \tag{12.15}$$

Introducing the value of C from (6.15) into (12.12)

$$C = \frac{8\pi^3(\varepsilon^* - \varepsilon)}{k^3 ci|\varepsilon - 1|^2} I_{0\omega} \approx \frac{8\pi^3(\varepsilon^* - \varepsilon)}{k^3 ci|\varepsilon|^2} I_{0\omega}$$

and using (12.14), we obtain the following value for the constant $\mathcal{C}$:

$$\mathcal{C} = \frac{8\pi^3(\sqrt{\varepsilon^*\mu} + \sqrt{\varepsilon\mu})}{k^2 c|\varepsilon|} I_{0\omega} = \frac{8\pi^3\mu d}{kc} I_{0\omega}. \tag{12.16}$$

In the domain of applicability of the Rayleigh-Jeans law, when $I_{0\omega} = \frac{\theta k^2}{4\pi^3}$, we have

$$\mathcal{C} = \frac{2\theta}{c}\,\frac{\sqrt{\varepsilon^*\mu} + \sqrt{\varepsilon\mu}}{|\varepsilon|} = \frac{2\theta}{c} k\mu d. \tag{12.17}$$

In conclusion, let us emphasize the complete analogy between the methods of introducing the lateral fluctuating field in different cases. In all cases this field is introduced into the corresponding form of the

generalized Ohm's law.

In the differential volume form of this law we let

$$\vec{j} = \frac{(\varepsilon - 1)i\omega}{4\pi}(\vec{E} + \vec{K}),$$

where $\frac{(\varepsilon - 1)i\omega}{4\pi}$ is the complex conductance and where, in the absence of external macroscopic sources (i.e., $\vec{K}$ is the potential only of the fluctuating field),

$$\overline{\vec{K}} = 0, \quad \overline{K_\alpha(\vec{r}')K^*_\beta(\vec{r}'')} = C\delta_{\alpha\beta}\,\delta(x' - x'')\delta(y' - y'')\delta(z' - z''). \tag{12.18}$$

The boundary conditions (11.7) can, according to (11.2), be rewritten in the same way in the differential form of Ohm's law, but for the surface current $\vec{i}$

$$\vec{i} = \frac{c}{4\pi}\sqrt{\frac{\varepsilon}{\mu}}(\vec{E}_t + \vec{\mathcal{K}}).$$

Thus, $\frac{c}{4\pi}\sqrt{\frac{\varepsilon}{\mu}}$ is the surface conductivity, and for the surface fluctuating field $\mathcal{K}$ we have

$$\overline{\vec{\mathcal{K}}} = 0; \quad \overline{\mathcal{K}_\alpha(\vec{r}')\mathcal{K}^*_\beta(\vec{r}'')} = \mathcal{C}\delta_{\alpha\beta}\,\delta(x' - x'')\delta(y' - y'').$$

Finally, for quasi-stationary currents in networks consisting of quasi-linear (sufficiently thin) conductors, the starting-point is the integral form of the generalized Ohm's law, into which is introduced the lateral integral fluctuating emf $\mathcal{E}_{lat}$

$$I = \frac{\mathcal{E} + \mathcal{E}_{lat}}{Z}.$$

Here $1/Z$ is the admittance of the network, the mean value of $\mathcal{E}_{lat.}$ is equal to zero, the lateral emf in the non-overlapping parts of the network are not correlated between themselves, and the spectral intensity of the

integral random emf (for positive frequencies) is given by Nyquist's formula, deducible from (12.18) (see Chapter V)

$$\overline{\mathcal{E}_{lat}} = 0, \qquad \overline{2|\mathcal{E}_{lat}|^2} = \frac{2\theta}{\pi}\,\mathrm{Re}\,Z\,.$$

Section 13. Radiation of an Infinite Plane

The problem to be studied here is a particular case of the problem of radiation of a medium, occupying a half-space (section 6), but with the condition that for this medium $|\varepsilon| \gg 1$ The solution for this particular case, based on the application of the boundary conditions (11.7), permits a more accurate clarification of the influence on the result of the transition from conditions (11.7), which for the components assume the form

$$\left.\begin{aligned} \sqrt{\varepsilon}\,E_1 + \sqrt{\mu}\,H_2 &= -\sqrt{\varepsilon}\,\mathcal{K}_1\,, \\ \sqrt{\varepsilon}\,E_2 - \sqrt{\mu}\,H_1 &= -\sqrt{\varepsilon}\,\mathcal{K}_2 \end{aligned}\right\} \quad (z = 0), \qquad (13.1)$$

to the limiting form (11.8)

$$E_1 = -\mathcal{K}_1, \qquad E_2 = -\mathcal{K}_2\,. \qquad (13.2)$$

We must now introduce the field, of direct interest to us, in the half-space $z > 0$ (vacuum). Expanding the surface lateral field $\mathcal{K}$ into a double Fourier integral (12.1), and the potentials $\vec{E}$ and $\vec{H}$ in vacuum according to plane waves [expansion (12.4)] and sbustituting both expansions into condition (13.1), we obtain two equations for component $\vec{v}$

$$(t_3\sqrt{\mu} - k\sqrt{\varepsilon})\,v_\alpha - p_\alpha\sqrt{\mu}\,v_3 = k\sqrt{\varepsilon}\,g_\alpha \qquad (\alpha = 1,2). \quad (13.3)$$

The transverse wave condition constitutes the third equation

$$\vec{t}\vec{v} = p_1v_1 + p_2v_2 + t_3v_3 = 0\,. \qquad (13.4)$$

Solving these equations for $\vec{v}$, we get

$$v_\alpha = \sqrt{\varepsilon}\,\frac{k(t_3\sqrt{\varepsilon} - k\sqrt{\mu})g_\alpha + p_\alpha\sqrt{\mu}\,(p_1 g_1 + p_2 g_2)}{(t_3\sqrt{\varepsilon} - k\sqrt{\mu})(t_3\sqrt{\mu} - k\sqrt{\varepsilon})} \qquad (\alpha = 1,2),$$

$$v_3 = \sqrt{\varepsilon}\,\frac{p_1 g_1 + p_2 g_2}{t_3\sqrt{\varepsilon} - k\sqrt{\mu}} \tag{13.5}$$

using, further, the correlation function (12.11) for g, we find

$$\overline{\vec{v}\vec{v}^*} = \frac{C|\varepsilon|}{(2\pi)^2}\delta(p_1 - p_1')\delta(p_2 - p_2')\left\{\frac{k^2}{|t_3\sqrt{\mu} - k\sqrt{\varepsilon}|^2} + \right.$$

$$\left. + \frac{p_1^2 + p_2^2 + |t_3|^2}{|t_3\sqrt{\varepsilon} - k\sqrt{\mu}|^2}\right\}. \tag{13.6}$$

The current density along the z-axis is expressed by the formula which follows from (1.6) and (13.4)

$$S_{\omega 3} = -\frac{c}{4\pi k}\iint_{-\infty}^{+\infty} e^{i(t_3 - t_3^*)z}(t_3 + t_3^*)\overline{(\vec{v}, \vec{v}^*)}\, dp_1 dp_2 dp_1' dp_2'$$

Substituting in here (13.6) and taking into account that $t_3 + t_3^* \neq 0$ only when $p_1^2 + p_2^2 \leqslant k^2$, i.e., for real t_3, we obtain for $S_{\omega 3}$ the form

$$S_{\omega 3} = -\frac{kc\,C|\varepsilon|}{(2\pi)^3}\int_{p_1^2+p_2^2\leqslant k^2} t_3\left\{\frac{1}{|t_3\sqrt{\mu} - k\sqrt{\varepsilon}|^2} + \frac{1}{|t_3\sqrt{\varepsilon} - k\sqrt{\mu}|^2}\right\}dp_1 dp_2 .$$

Transforming into polar coordinates

$$p_1 = k\sin\theta\cos\varphi, \quad p_2 = k\sin\theta\sin\varphi, \quad t_3 = -k\cos\theta,$$

$$dp_1 dp_2 = k^2\cos\theta\sin\theta\, d\theta\, d\varphi = k^2\cos\theta\, d\Omega$$

then gives

$$s_{\omega 3} = \frac{k^2 c\, \mathcal{C} |\varepsilon|}{(2\pi)^3} \int_{\theta<\pi/2} \cos^2\theta \left\{ \frac{1}{|t_3\sqrt{\mu} - k\sqrt{\varepsilon}|^2} + \frac{1}{|t_3\sqrt{\varepsilon} - k\sqrt{\mu}|^2} \right\} d\Omega. \tag{13.7}$$

Finally, the comparison of this expression with (5.7) gives the intensity in the direction forming the angle θ with the normal. In order to convert the formula for the intensity to a form more suitable for comparison with (I.15) -- the expression for the radiation intensity of the half-space filled with the medium, we shall directly substitute the value (12.12) for $\mathcal{C}$. We finally obtain

$$I_\omega = \frac{k^3 c|\varepsilon|^2 lC}{8\pi^3} \cdot \frac{\cos\theta}{\sqrt{\varepsilon^*\mu} - \sqrt{\varepsilon\mu}} \left\{ \frac{\mu^2}{|\mu\cos\theta + \sqrt{\varepsilon\mu}|^2} + \frac{|\varepsilon|\mu}{|\varepsilon\cos\theta + \sqrt{\varepsilon\mu}|^2} \right\} \tag{13.8}$$

while the general expression (I.15) is

$$I_\omega = \frac{k^3 c|\varepsilon - 1|^2 lC}{8\pi^3} \cdot \frac{\cos\theta}{\zeta^* - \zeta} \left\{ \frac{\mu^2}{|\mu\cos\theta + \zeta|^2} + \frac{\sin^2\theta + |\zeta|^2}{|\varepsilon\cos\theta + \zeta|^2} \right\} \tag{13.9}$$

$$\zeta = \sqrt{\varepsilon\mu - \sin^2\theta}\,.$$

The comparison of these two expressions shows that (13.8) does in fact correspond to the case $|\varepsilon| \gg 1$. Neglecting in (13.9) unity and $\sin^2\theta$ compared to ε and $\varepsilon\mu$, we obtain (13.8).

Let us now turn our attention to the following circumstance. When deriving (13.8) we started not with (13.2) but with the more precise boundary conditions (13.1), while the correlation function (12.11), which we used, had been obtained with the help of the limiting form of the boundary conditions (13.2). It would seem that in all cases of application of the correlation function (12.11) it is necessary in the determination of fields as well to be bound by the conditions (13.2). However, this is

not so. The fact is that the utilization of the more precise boundary condition (13.1) influences the evaluation of the correlation function to a higher degree with respect to $1/\sqrt{\varepsilon}$, than the evaluation of the field (and the intensity).

If we evaluate $\mathcal{Y}_{\alpha\beta}$ in the form of an expansion in inverse powers of $\sqrt{\varepsilon}$, utilizing conditions (13.1), then it happens that expression (12.11) remains valid up to terms of the order $1/\varepsilon$ with respect to the principal (taking terms of this order into account the form of $\mathcal{Y}_{\alpha\alpha}$ is complicated, while $\mathcal{E}_1$ and $\mathcal{E}_2$ prove to have been already correlated, i.e., $\mathcal{Y}_{12} \neq 0$).

On the other hand, if the expression for the intensity had been derived with the help of the boundary conditions (13.2), then we would have obtained

$$I_{\alpha} = \frac{k^3 c|\varepsilon|\mu\Theta}{8\pi^3} \cdot \frac{\cos\theta}{\sqrt{\varepsilon^*\mu} + \sqrt{\varepsilon\mu}}\left(1 + \frac{1}{\cos\varepsilon}\right).$$

This result follows from (13.8) by neglecting in the denominators (13.8) lesser terms with respect to $\sqrt{\varepsilon}$. In the second term of (13.8) such neglecting is valid only if

$$|\varepsilon| \cos\varepsilon \gg |\sqrt{\varepsilon\mu}|.$$

Irrespective of how large $|\varepsilon|$ may be, for angles ε close to $\pi/2$, this condition is violated.

Thus, the evaluation of I_{α} with the more precise boundary conditions (13.1) corresponds to taking into account the next order with respect to $1/\sqrt{\varepsilon}$, due to which a good approximation of the angular dependence of I for all angles ε is maintained. The correlation function (12.11), however, with an accuracy up to the correction of the order $1/\varepsilon$ remains unchanged, i.e., the same as the evaluation with the boundary conditions (13.2).

The problem of equilibrium radiation in a plane wave-guide, solved for the radiation from a volume source in section 7, is simplified to the

same extent as the problem of one-sided radiation. The equilibrium radiation, which is established in space between the ideal mirror and the radiating wall and which in the asymptotic case $\ell >> \lambda$ is uniform and isotropic, does not depend on the wall parameters. Therefore, any time we are interested specifically in equilibrium radiation (in any almost transparent medium; see section 8), we can take a body having a sufficiently high $|\epsilon|$ as the source of this radiation. The problem can then be formulated and solved with the help of the boundary conditions (13.1), i.e., without examination of the field in the interior of the radiating body. We shall later use this fact when establishing asymptotic laws for equilibrium radiation in anisotropic media (Chapter IV).

Section 14. Radiation of a Sphere

The problem concerning radiation of a sphere whose radius a is large compared to the skin-layer thickness d, but can be in any ratio whatever to the wavelength λ in the surrounding space (vacuum), has a series of facets not devoid in interest to us. In contrast to the cases of planes and cylinders, we are dealing here with a body bounded on all sides which in a certain manner influences the effect of "diffraction smoothing" of the field structure on the surface. Further, comparing the solution for the cylinder where we used an arbitrary ϵ with the solution for the sphere, we can readily convince ourselves of the extent of simplification involved in determining the external field for the case a >> d under consideration, when it is possible to utilize the boundary conditions (13.1). Finally the result, which permits (at least in principle) determination of the power emitted by the sphere for an arbitrary ratio of radius a to wavelength λ, gives a confirmation of the general expression (10.21).

It is sufficient for us to examine now the field exterior to the sphere (in vacuum), satisfying the uniform equations

$$\text{curl } \vec{E} = -ik\vec{H} \quad ; \quad \text{curl } \vec{H} = ik\vec{E}$$

since div $\vec{E} = 0$, only two fundamental vector functions are required : r

the representation not only of $\vec{H}$, but also of $\vec{E}$. These are[1]

$$\vec{M}_{mn} = z\left(\frac{im}{\sin\theta} P\,\vec{i}_2 - P'\vec{i}_3\right) e^{im\varphi} , \tag{14.1}$$

$$\vec{N}_{mn} = \left[\frac{n(n+1)}{\rho} zP\vec{i}_1 + \frac{1}{\rho}\frac{\partial \rho z}{\partial \rho}\left(P'\vec{i}_2 + \frac{im}{\sin\theta}P\vec{i}_3\right)\right] e^{im\varphi} ,$$

where, in accordance with the radiation conditions, we have

$$z = \sqrt{\frac{\pi}{2\rho}}\, H^{(2)}_{n+\frac{1}{2}}(\rho), \qquad \rho = kR \tag{14.2}$$

and also

$$P = P_n^{|m|}(\eta), \qquad \eta = \cos\theta, \qquad P' = \frac{\partial P_n^{|m|}}{\partial\theta}; \tag{14.3}$$

$\vec{i}_1$, $\vec{i}_2$ and $\vec{i}_3$ are unit vectors with respect to the coordinates R, θ and φ. For the function $\vec{M}$ and $\vec{N}$ the following relations are satisfied

$$\mathrm{curl}\,\vec{M} = k\,\vec{N} \quad ; \quad \mathrm{curl}\,\vec{N} = k\,\vec{M} .$$

Omitting for brevity in writing the indices m and n in the fundamental functions as well as in the constants we have the following solution:

$$\vec{E} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} (A\vec{M} + B\vec{N}),$$
$$\vec{H} = i\sum_{n=0}^{\infty}\sum_{m=-n}^{n} (A\vec{N} + B\vec{M}). \tag{14.4}$$

Let us note, that in waves with amplitude A, there does not exist a radial component $\vec{E}$. According to the terminology for the classification of waves and waveguides, these are transverse electrical or H-waves. They

1. J. A. Stratton, Theory of electromagnetism, section 7.11. We are again using complex functions ($e^{im\varphi}$) in lieu of real functions in cos mφ and sin mφ in Stratton.

correspond to emission of magnetic multi-fields. On the other hand, waves with amplitude B are transverse magnetic or E waves, emitted by electrical multi-fields.

Solution (14.4) must satisfy the boundary conditions (13.1)[1] on the surface of the sphere $R = a$ $(\rho = \alpha)$

$$\begin{aligned} \sqrt{\varepsilon}\, E_\theta + \sqrt{\mu}\, H_\varphi &= -\sqrt{\varepsilon}\, \mathcal{K}_\theta \\ \sqrt{\varepsilon}\, E_\varphi - \sqrt{\mu}\, H_\theta &= -\sqrt{\varepsilon}\, \mathcal{K}_\varphi . \end{aligned} \tag{14.5}$$

These conditions determine A and B in terms of the components of the surface lateral field $\mathcal{K}_\theta$ and $\mathcal{K}_\varphi$, for which we have a correlation function (12.13), which, in spherical coordinates, becomes

$$\mathcal{K}_\alpha(\theta, \varphi)\mathcal{K}^*_\beta(\theta', \varphi') = \mathcal{C}\, \delta_{\alpha\beta}\, \frac{\delta(\theta - \theta')\,\delta(\varphi - \varphi')}{a^2 \sin\theta}$$

$$(\alpha, \beta = \theta, \varphi) . \tag{14.6}$$

Solving with the help of (14.6) for $\overline{|A|^2}$ and $\overline{|B|^2}$, we find the emissive power of the sphere, which is expressed in terms of $\overline{|A|^2}$ and $\overline{|B|^2}$ as follows:

$$P_\omega = \frac{2c}{k^2} \sum_{n=1}^{\infty} \frac{n(n+1)}{2n+1} \sum_{m=-n}^{n} \frac{(n + |m|)!}{(n - |m|)!} \left(\overline{|A|^2} + \overline{|B|^2}\right) . \tag{14.7}$$

The mentioned evaluations are given in Appendix V. Their result is the following:

$$P_\omega = \frac{8\pi^2 I_{0\omega}}{k^5 a^2 d} \sum_{n=1}^{\infty} (2n + 1) \left\{ \frac{1}{\left|\sqrt{\varepsilon}\, z + \frac{1-/\mu}{} \frac{d\alpha z}{d\alpha}\right|^2} + \right.$$

1. In the particular case of a sphere conditions (14.5) are valid with a precision up to terms of the order of $\left(\frac{d}{a}\right)^2$ inclusive (S. M. Rytov, JETF 10, 180, 1940).

$$+ \frac{1}{\left| \frac{\sqrt{\varepsilon}}{\alpha} \frac{d\,\alpha z}{d\alpha} - i\sqrt{\mu}\, z \right|^2} \Bigg\} . \qquad (14.8)$$

If we introduce the absorption coefficients of H_{nm} and E_{nm}-waves, then P_{ω} takes the form

$$P_{\omega} = \frac{2\pi^2 I_{0\omega}}{k^2} \sum_{n=1}^{\infty} (2n + 1) \left(A_n^{(H)} + A_n^{(E)}\right) . \qquad (14.9)$$

The absorption coefficients do not depend on the index m so that summing with respect to m from -n to +n gives the multiplier (2n + 1) under the summation sign with respect to n. Therefore the summation in (14.9) in fact extends to all mutually orthogonal types of waves. We thus again obtain an expression of the form (10.21).

In examining the radiation of half-space or of a surface, we had seen that the travelling waves participating in creating the energy flow are specified merely by the sufficiently smooth harmonic components of the field on the surface of the emitting body -- by those whose period is greater than the wavelength in the surrounding space. This effect of "diffraction smoothing" takes place only for an infinitely extended structure: if the diffraction grid having a transmittance e^{ipx} is of finite dimensions then even in the case $p \gg k$ (very "thick" grid) travelling waves will be present. In the problem of cylinder radiation we had a mixed case. The infinite extension along the cylinder axis leads to a "diffraction smoothing" of the field structure in the direction of this axis: in formula (10.19) the summation extends only over the space wave numbers h, contained within the limits -k and +k. On the other hand, the finite transverse dimensions of the cylinder give the sum over all numbers n of the azimuthal harmony from $-\infty$ to $+\infty$, i.e., as many as one wishes parts of the field variation along the azimuth φ introduce their contribution into the radiation energy P_{ω}. In the case of the sphere, i.e., a body bounded on all sides, the "diffraction smoothing" is completely absent:

the sum, expressing P_ω, in principle extends over as high numbers and orders of tesseral harmonics as one wishes. Of course, the specific weight of very high harmonics is very small and factually n is limited (the more so, the smaller ka; see below), but a sharp boundary, obtained in the cases of planes and cylinders, is now absent.

As in the case of cylinder radiation, individual terms in (14.8), corresponding to different types of waves, are not physically distinguishable and only the entire sum taken as a whole is of interest. Let us clarify how the emissive power depends on $\alpha = ka$. We shall consider here that the sphere radius a varies at a fixed frequency ω. Let us introduce the parameter

$$\delta = kd\mu = (1 - i)\sqrt{\frac{\mu}{\varepsilon}} \qquad (14.10)$$

and let us suppose that the frequency ω lies within the limits of applicability of the Rayleigh-Jeans law. Then formula (14.8) gives the following expression for the power emitted from a unit surface of the sphere

$$p_\omega = \frac{P_\omega}{4\pi a^2} = \frac{\theta k^2 \delta}{2\pi^2} \frac{1}{\alpha^2} \sum_{n=1}^{\infty} (2n + 1) \cdot$$

$$\cdot \left\{ \frac{1}{|(1 - i)\alpha z + i\delta(\alpha z)'|^2} + \frac{1}{|(1 - i)(\alpha z)' - i\delta\alpha z|^2} \right\} \qquad (14.11)$$

The smaller α, the faster does the series converge. For $\alpha \ll 1$ (but, at the same time $\alpha \gg \delta$, since $a \gg d$), practically only the first term in flower brackets for n = 1 is different from zero. This term corresponds to emission of a magnetic dipole and is equal to $\alpha^2/2$. Thus,

$$p_\omega = \frac{3\theta k^2 \delta}{4\pi^2}, \qquad P_\omega = \frac{3\theta\delta\alpha^2}{\pi} \quad (\delta \ll \alpha \ll 1). \qquad (14.12)$$

The predominance, in the case of a small sphere, of the emission of the

magnetic dipole over the emission of the electric dipole is explained by the fact that on the surface of the sphere are imposed not currents but electropropulsive forces (lateral field $\mathcal{K}$).

As α increases, the number of terms in the series of appreciable size increases rapidly: already for $\alpha = 3$ it is necessary to take terms up to n = 6 into account. For $\alpha \gg 1$ the number of important terms in the sum reaches a value of the order of α. For the approximate estimation of the asymptotic value (14.11) we shall proceed as follows: for $z = \sqrt{\frac{\pi}{2\alpha}} H^{(2)}_{n+1/2}(\alpha)$ let us use the asymptotic expression

$$\alpha z \approx i(\alpha z)' \approx e^{-i\left[\alpha - \frac{(n+1)\pi}{2}\right]}$$

and let us expand the summation up to $n = \chi\alpha$, where $\chi \sim 1$. Then

$$P_\omega \approx \frac{\theta k^2 \delta}{2\pi^2 \alpha^2} \sum_{n=1}^{\chi\alpha} (2n+1)\left\{\frac{1}{|1-\delta-i|^2} + \frac{1}{|1+\delta-i|^2}\right\} \approx$$

$$\approx \frac{\theta k^2 \delta}{\pi^2 \alpha^2} \sum_{n=1}^{\chi\alpha} n \approx \frac{\chi^2 \theta k^2 \delta}{2\pi^2} .$$

The evaluation based on Kirchhoff's law (see section 15) gives in the limiting case under consideration

$$P_\omega = \frac{2\theta k^2 \delta}{3\pi^2} .$$

Thus, to improve the accuracy of the result of the above estimate, we must let $\chi^2 = 4/3$, i.e., $\chi = 1.155$.

In Fig. 3 are shown the dependence on α of the quantities

$$P = P_m + P_e$$

$$p_m = \frac{1}{\alpha^2}\sum_{n=1}\frac{2n+1}{|(1-i)\alpha z + i\delta(\alpha z)'|^2}$$

$$p_e = \frac{1}{\alpha^2}\sum_{n=1}\frac{2n+1}{|(1-i)(\alpha z)' - i\delta\alpha z|^2} .$$

According to (14.11), these quantities represent to a definite scale the unit (per unit surface) radiation intensities -- p_m gives the radiation of magnetic multi-fields and p_e that of electric multi-fields. The curves have been constructed up to $\alpha = 2.5$ for $\delta \leqslant 0.001$. Such small δ's begin to affect the values of p_m and p_e only for $\alpha > 3$, so that the evaluations have been carried out according to the rather simple formulae

$$p_m = \frac{1}{2\alpha^4}\sum_{n=1}\frac{2n+1}{|z|^2}, \qquad p_e = \frac{1}{2\alpha^2}\sum_{n=1}\frac{2n+1}{|(\alpha z)'|^2} .$$

As α increases, the intensity of magnetic emission of the multi-fields drops monotonously, that of the electric first increases and then drops, passing through weakly expressed maxima somewhat shifted to the right from the position of its own values for electric oscillations of a sphere.[1] Starting with $\alpha \approx 0.75$, p_e exceeds p_m. The total unit emissive power $p = p_m + p_e$ approaches the value $p_\infty = \chi^2 = 4/3$ as $\alpha \to \infty$. Thus, for a small sphere ($\alpha \sim 1$) the emission from a unit surface is one-and-a-half to two times the intensity of a large sphere ($\alpha \gg 1$).

The approximate boundary conditions (13.1) and (13.2) are very often applicable in the radio-band of very high frequencies, since the skin-effect is here sufficiently well developed even for vanishing conductivities. An important particularity of most of the problems in this field is the fact that the propagation of waves does not occur in free space but in feeders limited by metallic walls -- of wave-guides or co-axial lines. Using conditions (13.1), let us examine some questions

1. See J. A. Stratton, Theory of electromagnetism, p. 489.

concerning thermal radiation in the ultra high frequency band.

Section 15. Radiation of a Partition in a Wave Guide

Let a flat partition (Fig. 4) be placed into the cylinder wave-guide, parallel to the x-axis at the z = 0 section. The walls of the wave-guide will at first be assumed ideally conducting so that only the partition constitutes a radiation source. We shall further assume that the skin-effect in the material of the partition is so strongly developed for the frequencies ω under study that the boundary conditions (13.1) can be utilized. The form of the cross-section of the wave guide is limited only by the requirement that it constitute a finite, continuous domain.

Let the potentials of $\vec{E}$ and $\vec{H}$ of the electric and magnetic fields in the wave-guide be proportional to $e^{-i\beta z}$. As is known,[1] the complete system of natural waves can be represented by the joining of waves of two types: 1) electrical or transverse magnetic (E-waves), for which $H_z = 0$ and $E_z = \Phi(x,y)e^{-i\beta z}$, where Φ is some kind of a natural function of the boundary problem

$$\Delta_2\Phi + \varkappa^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \varkappa^2\Phi = 0, \qquad (15.1)$$

$$\varkappa^2 = k^2 - \beta^2, \qquad (15.2)$$

where the condition

$$\Phi = 0 \qquad (15.3)$$

is imposed on the contour Γ of the cross-section of the wave-guide, 2) magnetic or transverse-electric (H-waves), for which $E_z = 0$, and $H_z = \Psi(x,y)e^{-i\beta z}$, where Ψ is the natural function of the boundary problem, corresponding to the same differential equation (15.1) but with

1. See, for instance, B. A. Vedenskii and A. G. Arenberg, Radio Wave-guides, Part I, (M. - L., 1946).

the condition on the contour Γ

$$\frac{\partial \Psi}{\partial n} = 0 \qquad (15.4)$$

where n is the normal to the contour.

Let us denote the natural values and the natural functions for E-waves, respectively, by $\varkappa_{mn}$ and Φ_{mn}, and for H-waves by $\varkappa'_{mn}$ and Ψ_{mn}. All natural functions are orthogonal between themselves whose normalization (both for Φ and Ψ) we shall make such that

$$\frac{1}{S}\int_S \Phi_{mn}\Phi^*_{\mu\nu}\,dx\,dy = \delta_{m\mu}\delta_{n\nu} \qquad (15.5)$$

where S is the cross-sectional area.

The general solution, consisting only of those waves which propagate or attenuate in the positive direction of the axis z, is then written in the form

$$E_x = \sum_{m,n}\left(A\frac{\beta}{i\varkappa^2}\frac{\partial\Phi}{\partial x}e^{-i\beta z} + B\frac{k}{i\varkappa'^2}\frac{\partial\Psi}{\partial y}e^{-i\beta' z}\right)$$

$$E_y = \sum_{m,n}\left(A\frac{\beta}{i\varkappa^2}\frac{\partial\Phi}{\partial y}e^{-i\beta z} - B\frac{k}{i\varkappa'^2}\frac{\partial\Psi}{\partial x}e^{-i\beta' z}\right)$$

$$E_z = \sum_{m,n} A\Phi e^{-i\beta z} \qquad (15.6)$$

$$H_x = \sum_{m,n}\left(-A\frac{k}{i\varkappa^2}\frac{\partial\Phi}{\partial y}e^{-i\beta z} + B\frac{\beta'}{i\varkappa'^2}\frac{\partial\Psi}{\partial x}e^{-i\beta' z}\right)$$

$$H_y = \sum_{m,n}\left(A\frac{k}{i\varkappa^2}\frac{\partial\Phi}{\partial x}e^{-i\beta z} + B\frac{\beta'}{i\varkappa'^2}\frac{\partial\Psi}{\partial y}e^{-i\beta' z}\right)$$

$$H_z = \sum_{m,n} B\Psi e^{-i\beta' z}\,.$$

For simplicity in writing we have here omitted the indices m, n for the functions Φ, Ψ, the quantities β, $\varkappa$, β', $\varkappa'$, and the constants A, B.

An easy evaluation (see Appendix VI) gives the following expression for the power crossing the cross-section of the wave-guide.

$$P_{\omega} = \frac{kcS}{2\pi} \sum_{m,n}{}' \left\{ \frac{\beta}{\varkappa^2} \overline{|A|^2} + \frac{\beta'}{\varkappa'^2} \overline{|B|^2} \right\}. \qquad (15.7)$$

The prime on the summation sign indicates that summation extends only to travelling ("up-to-critical") waves, i.e., only to those values of m and n for which β and β' are real ($\varkappa < k$ and $\varkappa' < k$). Thus, it is necessary to evaluate $\overline{|A|^2}$ and $\overline{|B|^2}$, which determine the intensity of individual E and H waves, for the evaluation of P_{ω}. To this end one must utilize the boundary conditions (13.1) on the surface of the plate

$$\begin{aligned} \sqrt{\varepsilon}\, E_x + \sqrt{\mu}\, H_y &= \sqrt{\varepsilon}\, \mathcal{K}_x \\ & \qquad (z = 0) \qquad (15.8) \\ \sqrt{\varepsilon}\, E_y - \sqrt{\mu}\, H_x &= -\sqrt{\varepsilon}\, \mathcal{K}_y \end{aligned}$$

and the correlation function (12.13)

$$\overline{\mathcal{K}_{\alpha}(x,y)\mathcal{K}^{*}_{\beta}(x',y')} = \mathcal{C}\, \delta_{\alpha\beta}\, \delta(x-x')\delta(y-y') \qquad (\alpha, \beta = x, y).$$
(15.9)

This evaluation, carried out in Appendix VI, gives the following result:

$$\overline{|A|^2} = \frac{\mathcal{C}\, |\varepsilon|\, \varkappa^2}{S\,|k\sqrt{\mu} + \beta\sqrt{\varepsilon}\,|^2}, \qquad \overline{|B|^2} = \frac{\mathcal{C}\, |\varepsilon|\, \varkappa'^2}{S\,|\beta'\sqrt{\mu} + k\sqrt{\varepsilon}\,|^2}. \qquad (15.10)$$

Inserting (15.10) into (15.7) and substituting the value (12.17) for $\mathcal{C}$ (we immediately limit ourselves to the non-quantum domain $\hbar\omega \ll \theta$), we obtain

$$P_{\omega} = \frac{\theta k}{\pi}\left(\sqrt{\varepsilon^{*}\mu} + \sqrt{\varepsilon\mu}\right)\sum_{m,n}'\left(\frac{\beta}{|k\sqrt{\mu}+\beta\sqrt{\varepsilon}|^{2}} + \frac{\beta'}{|\beta'\sqrt{\mu}+k\sqrt{\varepsilon}|^{2}}\right) \tag{15.11}$$

Since [see (15.2) and (14.10)]

$$\beta = k\sqrt{1-\left(\frac{\varkappa}{k}\right)^{2}}, \quad \beta' = k\sqrt{1-\left(\frac{\varkappa'}{k}\right)^{2}}, \quad \sqrt{\frac{\varepsilon}{\mu}} = \frac{1-i}{\delta}$$

(15.11) can similarly be rewritten in the form

$$P_{\omega} = \frac{2\theta}{\pi\delta}\sum_{m,n}'\left\{\frac{\sqrt{1-\left(\frac{\varkappa}{k}\right)^{2}}}{\left|1+\frac{1-i}{\delta}\sqrt{1-\left(\frac{\varkappa}{k}\right)^{2}}\right|^{2}} + \frac{\sqrt{1-\left(\frac{\varkappa'}{k}\right)^{2}}}{\left|\sqrt{1-\left(\frac{\varkappa'}{k}\right)^{2}}+\frac{1-i}{\delta}\right|^{2}}\right\}. \tag{15.12}$$

It is not difficult to understand the similarity of formula (15.11) and expression (13.7) for the density of energy flow from an infinite plane. Evidently, in the limiting case of wave-lengths λ, so small compared to the cross-sectional dimensions of the wave guide that the approximation of geometric optics can be assumed to be valid, the emission of the partition must coincide with the power flow which would be sent out into free half-space by a plate of the same area, material and temperature as for the partition. Let us convince ourselves of this.

Within the approximation of geometrical optics, we can evaluate the integral radiation from the plate of area S into half-space by simply multiplying (13.7) by S. By further substituting the value of $\mathcal{C}$ from (12.17) into (13.7), integrating with respect to the azimuth φ ($d\Omega = \sin\theta\, d\theta\, d\varphi$), which evidently is equivalent to multiplying by 2π, and using relation (14.10), we obtain the following expression for the integral flow of power from the plate to half-space:

$$P_{\omega\,pl.} = \frac{\theta k^{2}S}{\pi^{2}\delta}\int_{0}^{\pi/2}\left\{\frac{1}{\left|1+\frac{1-i}{\delta}\cos\theta\right|^{2}} + \frac{1}{\left|\cos\theta+\frac{1-i}{\delta}\right|^{2}}\right\}\cos^{2}\theta\,\sin\theta\, d\theta\,. \tag{15.13}$$

The first part corresponds to waves for which the electric vector is parallel to the plane of incidence, the second to waves for which this vector is perpendicular to the plane of incidence. In the first case there exists a normal to the plane of the plate constituting the vector $\vec{E}$, in the second - the vector $\vec{H}$.

Approaching now the examination, under the same conditions (small λ), of radiation of the partition in the wave guide, we shall take, for simplicity, the particular case of a rectangular wave guide. If a and b are the sides of the rectangle, then[1]

$$\left(\frac{\varkappa_{mn}}{k}\right)^2 = \left(\frac{\varkappa'_{mn}}{k}\right)^2 = \frac{\lambda^2}{4}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right). \qquad (15.14)$$

For $\lambda \ll$ a and b one can, in (15.12), go from summation to integration

$$\sum_{m,n}{}' \quad \iint_0^{\varkappa = k} dm\, dn$$

Further, let us introduce new integration variables θ and φ instead of m and n

$$\frac{\lambda m}{2a} = \sin\theta\cos\varphi, \qquad \frac{\lambda n}{2b} = \sin\theta\sin\varphi.$$

For these variables

$$\sqrt{1 - \left(\frac{\varkappa_{mn}}{k}\right)^2} = \cos\theta, \qquad dm\, dn = \frac{k^2 S}{\pi^2}\cos\theta\sin\theta\, d\theta\, d\varphi\,.$$

Thus, integration on the plane (m,n) over the quarter ellipse with poles $\frac{2a}{\lambda}$ and $\frac{2b}{\lambda}$ located in the first quadrant is reduced to integration from 0 to $\frac{\pi}{2}$ for φ and from $\frac{\pi}{2}$ to 0 for θ. In the result, after integration with respect to φ (i.e., multiplication by 2π) formula (15.12)

1. B. A. Vvedenskii and A. G. Arenberg, Radio waveguides, Part I.

gives exactly the same expression (15.13) as for the radiation of a plate into free half-space. The first part of (15.13) corresponds thus to E-waves, the second to H-waves. This is understandable, if we take into account the previously mentioned disposition of the vector $\vec{E}$ in the waves of both polarizations.

Keeping in mind that $\delta = k\mu d \ll 1$, the integral entering (15.13) can be evaluated by discarding in the denominators terms not containing $\frac{1-i}{\delta}$. Then the integral equals $\frac{2\delta^2}{3}$ and we obtain

$$P_\omega = \frac{2\Theta}{3\pi^2} k^2 \mathcal{E} S = \frac{2\Theta}{3\pi^2} k^3 \mu \, dS \; . \tag{15.15}$$

It is this expression that we had referred to above when we carried out the estimation of thermal radiant power of a cylinder and sphere for the asymptotic case of very small wavelengths.

The simplification made by us in the expression under the integral sign in (15.13) which means that in (15.12) we preserve only the lowest (first) degree in δ and in (15.11) -- the highest in $\sqrt{\varepsilon}$, corresponds to the transition to the limiting form of the boundary conditions (15.8), i.e., to the conditions

$$E_x = -\mathcal{H}_x, \quad E_y = -\mathcal{H}_y \quad (z = 0). \tag{15.16}$$

With the aid of the boundary conditions of Leontovich (11.6) it is not difficult to find (see Appendix VI) the absorption coefficients of the partition for the waves E_{mn} and H_{mn} to be, respectively,

$$A_{mn} = \frac{2k(\sqrt{\varepsilon^*\mu} + \sqrt{\varepsilon\mu})\beta_{mn}}{|\beta_{mn}\sqrt{\varepsilon} + k\sqrt{\mu}|^2} ; \qquad A'_{mn} = \frac{2k(\sqrt{\varepsilon^*\mu} + \sqrt{\varepsilon\mu})\beta'_{mn}}{|\beta'_{mn}\sqrt{\mu} + k\sqrt{\varepsilon}|^2} . \tag{15.17}$$

Therefore, expression (15.11) for the power, which the partition emits into the waveguide at a frequency ω, can be written in the form

$$P_\omega = \frac{\theta}{2\pi} \sum_{m,n}' (A_{mn} + A'_{mn}) \qquad (15.18)$$

i.e., the expression is reduced to the same form (10.21) for P_ω for the case of a cylinder or sphere in a free space. A particularity of the given problem is the finite number of mutually orthogonal travelling waves, which becomes smaller as the dimensions of the waveguide cross-section compared to the wavelength becomes smaller. However, the general formulation of the position (10.21) does not solve the question whether the union of the indicated waves is finite or infinite. Of importance is only the fact that discussion concerns travelling waves (waves participating in the transport of energy). To this is related the possibility of obtaining (15.18) on the basis of classical radiation theory (section 17), as we shall see later.

Let us also note the following. In this chapter, all derivations are limited by the assumption $|\varepsilon| \gg 1$, i.e., they refer only to strongly reflecting bodies (small absorption coefficients A). However, expression (15.18) is not restricted in any way to size-limitations of the coefficients A which to a great extent appears to be evident but will be proven later. Formula (15.18) remains, thus, valid for any degree of agreement, also when any of the absorption coefficients is equal to one. Some confirmation of this can be obtained in the analogous form of the result which had been obtained before in the radiation problem of a cylinder made of a material with arbitrary electric parameters.

In conclusion let us examine the utility of formula (15.18) in the region of rather practical interest for which the wavelength λ is comparable to the dimensions of the waveguide cross-section.

If, for a given frequency ω, only one wave type, for instance H_{10}, is subcritical, then the power emitted on this wave in the frequency interval $d\omega$ will be

$$\frac{\theta}{2\pi} A'_{10}\, d\omega = kTA'_{10}\, df \qquad (\omega = 2\pi f) .$$

In the case of a concordant radiator ($A'_{10} = 1$) this power is equal to

$$kT\, df\,. \qquad (15.19)$$

Let us clarify how the summed absorption coefficient $A = \sum_{m,n}' (A_{mn} + A'_{mn})$ in the function varies with the frequency ω. In order to fix all characteristics it is sufficient to limit one-self to any special form of the cross-section of the waveguide. Let us take the rectangular waveguide with $a = 2b$. Introducing the parameter

$$\xi = \frac{2a}{\lambda} = \frac{a\omega}{\pi c}$$

we obtain

$$\sqrt{1 - \left(\frac{\chi_{mn}}{k}\right)^2} = \sqrt{1 - \left(\frac{\chi'_{mn}}{k}\right)^2} = \frac{1}{\xi}\sqrt{\xi^2 - (m^2 + 4n^2)}$$

$$\delta = k\mu d = \sqrt{\xi}/\gamma, \quad \text{where} \quad \gamma = \sqrt{\frac{2a\sigma}{c\mu}}\,.$$

For copper with $a = 2$ cm we have $\gamma = 8400$ so that $\delta \ll 1$ right up to $\xi \sim 10^6$.

Both A_{mn} and A'_{mn} enter the game, starting with a zero value for $\xi^2 = m^2 + 4n^2$. But the absorption coefficients of the H-waves (A'_{mn}) grow smoothly while the absorption coefficients of the E-waves (A_{mn}) have for $\xi^2 = \dfrac{m^2 + 4n^2}{1 - \xi/2\gamma^2}$ (practically - immediately after their appearance) a very pronounced maximum equal to $\dfrac{1}{1 + \sqrt{2}} = 0.83$. In view of the very large values $\dfrac{\sqrt{\xi}}{\gamma}$, the subsequent decrease in A_m with increasing ξ is described with great precision by the corresponding (mn) terms of formula (15.12) if in its denominator we discard unity. Concerning the coefficients A'_{mn} [the second terms in (15.12)]: in them, for any value of ξ, one can always discard in the denominator the first term. Thus, the approximate formula for A corresponding to the boundary

conditions (15.16), can be written in the form

$$A = \sum_{m,n}{}' (A_{mn} + A'_{mn}) =$$

$$= \frac{2\sqrt{\xi}}{\gamma} \sum_{m,n}{}' \left\{ \frac{\xi}{\sqrt{\xi^2 - (m^2 + 4n^2)}} + \frac{\sqrt{\xi^2 - (m^2 + 4\pi^2)}}{\xi} \right\} \quad (15.20)$$

where A_{mn} must be taken finite and equal to 0.83 when $\xi^2 = m^2 + 4n^2$.

Let us note that the difference in behavior in the absorption coefficients of E and H waves when varying the frequency is related to the fact that in the purely complex ε, i.e., in the material of the partition, only electric losses take place. For complex μ and real ε, the absorption coefficients A_{mn} will change smoothly with frequency, but sharp maxima will appear only for the coefficients A'_{mn}. In the general case, when ε and μ are complex, the picture will be rather complicated since the behavior of A_{mn} and of A'_{mn} will depend not only on the thickness of the skin-layer, but also on the ratio of losses of both forms.

In the rectangular waveguide for E waves not one of the indices m and n can be equal to zero, but for H waves only one of the indices can equal zero.[1] Therefore formula (15.20), upon writing out some of the first terms, has the following form

$$A = \frac{2\sqrt{\xi}}{\gamma} \left\{ \underset{(E_{11})}{\frac{\xi}{\sqrt{\xi^2 - 5}}} + \underset{(E_{21})}{\frac{\xi}{\sqrt{\xi^2 - 8}}} + \underset{(E_{31})}{\frac{\xi}{\sqrt{\xi^2 - 13}}} + \underset{(E_{12})}{\frac{\xi}{\sqrt{\xi^2 - 17}}} + \ldots + \right.$$

$$\left. + \underset{(H_{10})}{\frac{\sqrt{\xi^2 - 1}}{\xi}} + \underset{(H_{20})}{\frac{\sqrt{\xi^2 - 4}}{\xi}} + \underset{(H_{01})}{\frac{\sqrt{\xi^2 - 4}}{\xi}} + \underset{(H_{11})}{\frac{\sqrt{\xi^2 - 5}}{\xi}} + \underset{(H_{21})}{\frac{\sqrt{\xi^2 - 8}}{\xi}} + \ldots \right\}.$$

1. B. A. Vedenskii and A. G. Arenberg, Radio waveguides, Part I.

Under each of the fractions the wave type is indicated in brackets. Fig. 5 shows the dependence of $A\gamma$ on ξ for $\gamma = 1000$. Under the abscissa are indicated the wave types becoming "sub-critical" for corresponding values of ξ. The height of the sharp maxima equals 830. + value of the sum of the remaining terms for a given ξ. Thus, these maxima go far beyond the boundaries of the graph. The dotted line represents the curve

$$A\gamma = \frac{2\pi}{3}\xi^{5/2}$$

corresponding to the approximation of geometrical optics, i.e., to formula (15.15). In the sense of (15.15), this curve gives, for sufficiently large ξ, the value of $A\gamma$ averaged over the interval $\Delta\xi$ which, being highly monochromatic ($\Delta\xi \ll \xi$), includes at the same time a large number of the peaks of the exact curve $A\gamma$, i.e., a large number of wave types.

Section 16. Cylindrical resonator

In the previous section we were interested only in the energy flow, created in the waveguide by the "noisy" plate. This flow is entirely determined by travelling (sub-critical) waves and, consequently, if the cross-section of the waveguide is so small that at the frequency ω under study travelling waves are impossible, emission at the wavelength ω will be absent ($P_\omega = 0$). But the partition, besides creating the travelling waves, under all conditions also creates non-uniform (supra-critical) waves, which do not participate in energy transport and drop out of the expression for P_ω. The union of these non-uniform waves does not represent anything else but a thermal quasi-stationary field, completely analogous to the field of non-uniform waves in the previously studied cases of radiation of half space or of an infinite plane. As was already mentioned, everything concerning this quasi-stationary field lies completely outside the bounds of the classical radiation theory. On the other hand, the energy relations, connected with the travelling

waves, can, under certain circumstances, be obtained by classical methods.

It is therefore of a certain interest to study, from the point of view of the fluctuation theory being here developed, the question concerning the structure and energy of the thermal field of the "noise making" plate and, in particular, to examine the relation between the result (15.19) and the theory of energy distribution according to degrees of freedom, i.e., according to natural oscillations of the closed conservative system. For such a system we shall take a cylindrical resonator -- a segment of the waveguide between the partition and an ideally reflecting piston, located at some distance from the partition. Since the waveguide walls are assumed, as before, to be ideally conducting, such a resonator will be the closer to conservativeness, the higher the conductivity of the plate.

And thus the conditions of the previous problem are changed in the respect that at a distance ℓ from the partition is located an ideally conducting piston (Fig. 6). In the space between the piston and the partition a system of waves is established, which is made up of a superposition of "straight" waves (15.6) [travelling and non-uniform] and "counter" waves of analogous structure. The total potentials will be expressed by the sum (15.6) and the corresponding terms for the counter waves which differ from (15.6), firstly, by an opposite sign for β and β' and, secondly, by other constants: instead of A and B, we shall have, respectively, C and D.

The total field energy U_{ω}, contained in the resonator cavity, is

$$U_{\omega} = \int_V u_{\omega}\, dV = \int_0^{\ell} dz \int_S u_{\omega}\, dx\, dy \qquad (16.1)$$

where, according to (2.11)

$$u_{\omega} = \frac{1}{4\pi} \left\{ \overline{\vec{E}\vec{E}^*} + \overline{\vec{H}\vec{H}^*} \right\}. \qquad (16.2)$$

Since all the evaluations are basically a repetition of those carried out in the previous section and in Appendix VI, we shall give here only a

general description of the evaluational procedure.

Using the orthogonality conditions (15.5) and formulae (VI.1) and (VI.2), it can be shown that the energy density per unit length of resonator is

$$w_\omega = \int_S u_\omega dx\, dy =$$

$$= \frac{S}{4\pi} \sum_{m,n} \Big\{ \frac{2k^2 + \beta(\beta^* - \beta)}{\varkappa^2} [\, \overline{|A|^2}\, e^{i(\beta^* - \beta)z} + \overline{|C|^2}\, e^{-i(\beta^* - \beta)z} \,] +$$

$$+ \frac{2k^2 - \beta(\beta^* + \beta)}{\varkappa^2} [\, \overline{AC^*}\, e^{-i(\beta^* + \beta)z} + \overline{A^*C}\, e^{i(\beta^* + \beta)z} \,] +$$

$$+ \frac{2k^2 + \beta'(\beta'^* - \beta')}{\varkappa'^2} [\, \overline{|B|^2}\, e^{i(\beta'^* - \beta')z} + \overline{|D|^2}\, e^{-i(\beta'^* - \beta')z} \,] +$$

$$+ \frac{2k^2 - \beta'(\beta'^* + \beta')}{\varkappa'^2} [\, \overline{BD^*}\, e^{-i(\beta'^* + \beta')z} + \overline{B^*D}\, e^{i(\beta'^* + \beta')z} \,] \Big\} \qquad (16.3)$$

For brevity in writing, the indices m,n have again been dropped in β, $\varkappa$, β', $\varkappa'$ and in the constants A, B, C, D.

From the boundary conditions on the surface of the partition

$$\begin{aligned} \sqrt{\varepsilon}\, E_x + \sqrt{\mu}\, H_y &= -\sqrt{\varepsilon}\, \mathcal{H}_x \\ \sqrt{\varepsilon}\, E_y - \sqrt{\mu}\, H_x &= -\sqrt{\varepsilon}\, \mathcal{H}_y \end{aligned} \qquad (z = 0)$$

and on the surface of the piston

$$E_x = E_y = 0 \qquad (z = \ell).$$

The following values of the constants are determined:

$$C = Ae^{-2i\beta\ell} = -\frac{i\sqrt{\varepsilon}}{S\Delta} \int_S \left(\mathcal{H}_x \frac{\partial \Phi^*}{\partial x} + \mathcal{H}_y \frac{\partial \Phi^*}{\partial y} \right) dx\, dy$$

$$D = -B e^{-2i\beta'\ell} = \frac{i\sqrt{\varepsilon}}{S\Delta'}\int_S \left(\mathcal{K}_x \frac{\partial \Psi^*}{\partial y} - \mathcal{K}_y \frac{\partial \Psi^*}{\partial x}\right) dx\, dy$$

where

$$\begin{aligned} \Delta &= k\sqrt{\mu}\,(e^{2i\beta\ell} + 1) + \beta\sqrt{\varepsilon}\,(e^{2i\beta\ell} - 1) \\ \Delta' &= \beta'\sqrt{\mu}\,(e^{2i\beta'\ell} + 1) + k\sqrt{\varepsilon}\,(e^{2i\beta'\ell} - 1). \end{aligned} \tag{16.4}$$

From these expressions, using the correlation function (15.11), we find

$$\overline{|A|^2} = \overline{|C|^2}\, e^{2i(\beta - \beta^*)}, \quad \overline{AC^*} = \overline{|C|^2}\, e^{2i\beta\ell},$$

$$\overline{|C|^2} = \frac{\varkappa^2|\varepsilon|\,\mathcal{C}}{S|\Delta|^2}, \quad \overline{|B|^2} = \overline{|D|^2}\, e^{2i(\beta' - \beta'^*)\ell},$$

$$\overline{BD^*} = -\overline{|D|^2}\, e^{2i\beta'\ell}, \quad \overline{|D|^2} = \frac{\varkappa'^2|\varepsilon|\,\mathcal{C}}{S|\Delta'|^2}$$

Substitution of these values into (16.3) gives

$$\begin{aligned} v_\omega = \frac{|\varepsilon|\,\mathcal{C}}{2\pi}\sum_{m,n}\Bigg\{ & \frac{e^{-i(\beta^*-\beta)\ell}}{|\Delta|^2}\Big([2k^2 + \beta(\beta^* - \beta)]\cos(\beta^* - \beta)(z - \ell) + \\ & + [2k^2 - \beta(\beta^* + \beta)]\cos(\beta^* + \beta)(z - \ell)\Big) + \\ & + \frac{e^{-i(\beta'^* - \beta')\ell}}{|\Delta'|^2}\Big([2k^2 + \beta'(\beta'^* - \beta')]\cos(\beta'^* - \beta')(z - \ell) - \\ & - [2k^2 - \beta'(\beta'^* + \beta')]\cos(\beta'^* + \beta')(z - \ell)\Big)\Bigg\}. \end{aligned}$$

Let us now separate the terms with real values for β and β' (subcritical waves, for which $\varkappa$ and $\varkappa'$ are less than k) from the terms with imaginary values for β and β' (non-uniform waves, for which $\varkappa$ and $\varkappa'$

are greater than k), and for the latter we shall introduce the notation

$$\beta = i\alpha = i\sqrt{\varkappa^2 - k^2}, \qquad \beta' = i\alpha' = i\sqrt{\varkappa'^2 - k^2}. \qquad (16.5)$$

We obtain

$$w_\omega = w_{\omega\,waves} + w_{\omega\,quas},$$

$$w_{\omega\,waves} = \frac{|\mathcal{E}|\,\mathcal{C}}{\pi} \sum_{m,n}{}' \left\{ \frac{k^2 - \varkappa^2 \cos 2\beta(z-\ell)}{|\Delta|^2} + \frac{k^2 - \varkappa'^2 \cos 2\beta'(z-\ell)}{|\Delta'|^2} \right\}$$

$$w_{\omega\,quas} = \frac{|\mathcal{E}|\,\mathcal{C}}{\pi} \sum_{m,n}{}'' \left\{ \frac{k^2 + \varkappa^2 \cosh 2\alpha(z-\ell)}{|\delta|^2} - \frac{k^2 - \varkappa'^2 \cosh 2\alpha'(z-\ell)}{|\delta'|^2} \right\} \qquad (16.6)$$

where, according to (16.4)

$$\begin{aligned} |\Delta|^2 &= 4|k\sqrt{\mu}\cos\beta\ell + i\beta\sqrt{\varepsilon}\sin\beta\ell|^2, \\ |\Delta'|^2 &= 4|\beta'\sqrt{\mu}\cos\beta'\ell + ik\sqrt{\varepsilon}\sin\beta'\ell|^2, \\ |\delta|^2 &= 4|k\sqrt{\mu}\cosh\alpha\ell - i\alpha\sqrt{\varepsilon}\sinh\alpha\ell|^2, \qquad (16.7) \\ |\delta'| &= 4|i\alpha'\sqrt{\mu}\cosh\alpha'\ell - k\sqrt{\varepsilon}\sinh\alpha'\ell|^2. \end{aligned}$$

Let us examine first the linear energy density of the quasi-stationary field $w_{\omega quas}$. The presence of the ideally reflecting piston does not introduce here anything principally new and only complicates the picture. Since non-uniform waves rapidly weaken with withdrawal from the "noise producing" partition, we can return to the case of an unbounded waveguide, letting $\ell = \infty$. From (16.6) and (16.7) we then obtain

$$w_{\omega\,quas} = \frac{|\mathcal{E}|\,\mathcal{C}}{2\pi} \sum_{m,n}{}'' \left\{ \frac{\varkappa^2 e^{-2\alpha z}}{|k\sqrt{\mu} - i\alpha\sqrt{\varepsilon}|^2} + \frac{\varkappa'^2 e^{-2\alpha' z}}{|i\alpha'\sqrt{\mu} - k\sqrt{\varepsilon}|^2} \right. . \qquad (16.8)$$

Let us remind ourselves that the double prime on Σ means that summation

extends over those values of m and n, for which $\varkappa$ and $\varkappa'$ are greater than k. Thus, the sum (16.8) has no upper boundary and this permits, for estimating (16.8), to go over from summation to integration. The corresponding evaluation is done for a rectangular waveguide (which, of course, is immaterial) in Appendix VII and gives exactly the same result as was obtained before for the energy density of a quasi-stationary field in the case of an infinite plane boundary of a well conducting medium (Appendix I). In particular, for very large distances from the partition in the waveguide, when $kz \gg 1/\delta$, where $\delta = k\mu d \ll 1$, the approximate expression $w_{\omega\,quas}$ proves to be the same as for very small distances, when $kz \ll \delta$. In these two limiting cases

$$w_{\omega\,quas} \approx u_{0\omega}\,\frac{S}{4(kz)^2\delta} = u_{0\omega}\,\frac{S}{4k^3z^2\mu d} \qquad (kz \gg 1/\delta \text{ and } kz \ll \delta) \qquad (16.9)$$

In the domain $kz \ll \delta^3$ the increase in $w_{\omega\,quas}$ follows the law $\sim 1/z^3$ [this result, established in Appendix I, cannot be obtained in the approximation examined here, i.e., by use of the approximate boundary conditions (15.8)], and for distances comparable to the correlation radius a of the lateral field $\mathcal{H}$, the increase of $w_{\omega\,quas}$ as $z \to 0$ is limited by a value inversely proportional to a^3.

If the transverse dimensions of the resonator are so small that for the frequency ω under consideration travelling waves are not possible [the sum Σ' in (16.7) does not contain a single term], then only a thermal quasi-stationary field will be present at this frequency in the resonator. Assuming this case not to take place, let us now examine the first sum (16.7). The energy density of the sub-critical E and H waves entering it is not extinguished, but oscillates along the resonator axis (standing waves). Therefore, the total energy

$$U_{\omega\,waves} = \int_0^{\ell} w_{\omega\,waves}\,dz$$

contains a term proportional to the resonator length which is obtained

from the constants (independent of z) of the terms in Σ' and which is predominant at sufficiently high ℓ. It equals

$$U_\omega = \frac{|\varepsilon| \mathcal{C} k^2 \ell}{\pi} \sum_{m,n}' \left\{ \frac{1}{|\Delta|^2} + \frac{1}{|\Delta'|^2} \right\}$$

or, in view of (16.7),

$$U_\omega = \frac{|\varepsilon| \mathcal{C} k^2 \ell}{4\pi} \sum_{m,n}' \left\{ \frac{1}{|k\sqrt{\mu}\cos\beta\ell + i\beta\sqrt{\varepsilon}\sin\beta\ell|^2} + \right.$$

$$\left. + \frac{1}{|\beta'\sqrt{\mu}\cos\beta'\ell + ik\sqrt{\varepsilon}\sin\beta'\ell|^2} \right\}. \tag{16.10}$$

A change in ω causes the spectral density of the total energy U_ω to go through minima, alternating with maxima, which correspond to the natural frequencies of the resonator. The maxima are the higher, sharper and closer in frequency to the values

$$\beta_{mn}\ell = s\pi, \qquad \beta'_{mn}\ell = s\pi \qquad (s = 1,2,3,\ldots),$$

the smaller the attenuation in the plate, i.e., the closer the whole system is to being conservative.

Let us examine the behavior of U_ω in the vicinity of the resonance of any wave E_{mn}, i.e., in the vicinity of the maximum of the m,n-th term, standing in (16.10) at the first place. Since the wave type and the value of m and n are fixed, we shall for simplicity in writing omit as before the indices m,n. Let γ denote the quantity

$$\gamma = \frac{\beta}{k\delta} = \frac{1}{\delta}\sqrt{1 - \left(\frac{\varkappa}{k}\right)^2}\ .$$

Due to the smallness of $\delta = k\mu d$, the quantity γ is very large already at very small deviations of the studied wave from the critical frequency. If we use expression (12.1) for $\mathcal{C}$:

$$\mathcal{C} = \frac{8\pi^3\mu d}{kc} I_{0\omega} = \frac{2\pi^2\mu d}{k} u_{0\omega}$$

then the term of U_ω, corresponding to the wave E_{mn}, can be written in the form

$$\frac{\pi^2 \ell}{k^2 \delta} u_{0\omega} \frac{1}{|\cos \beta\ell + (1 + i)\gamma \sin \beta\ell|^2} . \qquad (16.11)$$

For large γ, the resonance maxima of this expression are so sharp that γ and δ can be considered as constants in their vicinity, and only the variation of the argument $\beta\ell$ of the sine and cosine need be taken into account. Maxima occur at frequencies for which

$$\sin 2\beta\ell \approx -1/\gamma, \quad \cos 2\beta\ell \approx 1 - 1/\gamma^2 .$$

We shall denote the resonance value of all quantities by the index 0. Let ω_0 be one of the resonance frequencies in whose vicinity we are interested. For brevity, let $\xi = \beta\ell$. Close to resonance, i.e., for small $\xi - \xi_0$, (16.11) becomes

$$\left(\frac{2\pi\ell u_{0\omega}}{k^2\delta}\right)_0 \frac{1}{1 + 4\gamma_0^2 (\xi - \xi_0)^2} .$$

In evaluating the energy, attributable to the natural standing wave of frequency ω_0, the last expression must be integrated with respect to ω across the width of the resonance maximum. In view of its sharpness it can in this matter expand the integration limits from $-\infty$ to $+\infty$. In the limit, in the transition to the final result as $|\varepsilon| \to \infty$ (ideally conducting partition, i.e., consecutive system) the error will tend to zero.

From the relation

$$\xi - \xi_0 = (\beta - \beta_0)\ell = (\sqrt{k^2 - \varkappa^2} - \sqrt{k^2 - \varkappa_0^2})\ell$$

we have

$$d\omega = \frac{c\beta_0}{k_0 \ell} d\xi .$$

Consequently, the energy, attributable to the natural wave, is equal to

$$\left(\frac{2\pi \ell u_{0\omega}}{k^2 \delta}\right)_0 \int_{-\infty}^{+\infty} \frac{d\omega}{1 + 4\gamma_0^2(\xi - \xi_0)^2} = \left(\frac{2\pi c \gamma u_{0\omega}}{k^2}\right)_0 \int_{-\infty}^{+\infty} \frac{d\xi}{1 + 4\gamma_0^2(\xi - \xi_0)^2} =$$

$$= \left(\frac{\pi c u_{0\omega}}{k^2}\right)_0 [\operatorname{arctg} 2\gamma_0(\xi - \xi_0)]_{-\infty}^{+\infty} = \frac{\pi^2 c}{k_0^2} u_{0\omega} . \quad (16.12)$$

In the general case of Planck's distribution (5.1) this gives the value

$$\varepsilon(\omega_0, T) = \hbar\omega_0\left(\frac{1}{2} + \frac{1}{e^{\hbar\omega/\theta} - 1}\right)$$

i.e., the mean energy of the oscillator of frequency ω_0 at temperature T and, correspondingly, in the domain $\hbar\omega_0 \ll \theta$, - at the value θ (theorem of equi-partition).

At sufficiently high frequencies, for which the wave lengths are much smaller than the resonator dimensions and, consequently, the approximations of geometrical optics are valid, it is not difficult to obtain the asymptotic value of U_ω. The corresponding evaluation, consisting of the averaging of (16.10) with respect to $\xi = \beta\ell$ and of the transition from summation to integration, is given in Appendix VII. It gives the obvious result

$$U_\omega = \ell S u_{0\omega} \quad (16.13)$$

i.e., the total energy is equal to the product of the resonator volume and the equilibrium energy density.

The results (16.12) and (16.13) refer to the equilibrium state and, evidently, do not depend on the parameters which characterize the material

of the "noisy" partition. These results remain valid even in the limiting case, when the partition is ideally conducting, so that the resonator represents a conservative system, deprived of emitting sources. The equilibrium state -- when it is established -- will also in this case correspond to (16.12) and (16.13). The question merely consists of this: how much time will be required for the establishment of equilibrium. This time can be estimated as follows.

Let us assume that the partition is covered by an ideally conducting screen and that in the resonator cavity, now entirely surrounded by ideally conducting walls, there is no radiation. Let the screen be removed at time $t = 0$. Radiation, representable by superposition of E and H waves, will begin to propagate from the partition. Let us examine the establishment of the E_{mn} wave. The phase and group velocities of this wave are, respectively,

$$v_{mn} = \frac{\omega}{\beta_{mn}} = \frac{c}{\sqrt{1 - \left(\frac{\varkappa_{mn}}{k}\right)^2}}, \quad v'_{mn} = \frac{d\omega}{d\beta_{mn}} = \frac{c\beta_{mn}}{k}. \tag{16.14}$$

The traversal of the wave E_{mn} from the partition to the piston and back requires the group time $\tau_{mn} = 2\ell/v_{mn}$. With each reflection at the plate the energy of the wave decreases in the ratio $(1 - A_{mn}):1$, where $A_{mn} = 2k\delta/\beta_{mn}$ is the limiting (for $|\varepsilon| \gg 1$) value of the energy absorption coefficient of this wave.

Let s be the number of traversals after which the wave energy becomes negligibly small compared to the initial energy, i.e., $(1 - A_{mn})^s \ll 1$. Evidently, it can be assumed that $s \sim 1/A_{mn}$. Thus, the time of establishment for the wave E_{mn} according to the order of the quantity will be:

$$T_{mn} = s\tau_{mn} \sim \frac{\tau_{mn}}{A_{mn}} = \frac{2\ell}{v_{mn}} : \frac{2k\delta}{\beta_{mn}} = \frac{\ell}{c\delta} = \frac{\ell}{c}\sqrt{\frac{2\pi\sigma}{\mu\omega}}.$$

This quantity does not depend on m and n, i.e., it is the same for all types and numbers of waves and can therefore be taken as the general time

of establishment of the equilibrium state. The order of T is the same as that of the time of establishment, determined by the mean quality of the resonator (without taking the wave type into account).[1]

Section 17. Waveguide Form of Kirchhoff's Law

We have seen that for a bounded and almost conservative system -- for a resonator -- a theorem is obtained in the asymptotic case $\mathcal{L} \gg \lambda$ [when in (16.6) one can neglect the interference oscillations of the energy density along the resonator axis z] concerning the distribution of energy according to the degrees of freedom in its quantum or classical form, depending on the relation of $\hbar\omega$ and θ. An important requirement for this result is the presence of sub-critical (travelling) waves, capable of forming in the closed space a system of standing waves, which, in turn, represent the superposition of the natural electromagnetic oscillations of the cavity under study. It is not difficult to understand, that, under the fulfillment of the indicated requirements, a reverse path is possible -- from the theorem concerning the energy distribution according to degrees of freedom to the law (10.2) for the emissive power

$$P_\omega = \frac{2\pi^2 I_{0\omega}}{k^2} \sum_i{}' A \qquad (17.1)$$

i.e., it is possible to derive this general expression on the basis of the classical theory of radiation, whereas this expression had previously been obtained by means of a straight electrodynamic evaluation of the radiation in a series of concrete special cases -- cylinder (section 10), sphere (section 14), partition in a waveguide (section 15). A classical derivation (17.1) of this type can be given if the initial derivation of Nyquist's formula is generalized in a corresponding manner.[2]

1. See J. Slater, Electronics of ultra high frequencies, p. 89, (Ed. "Soviet Radio", M., 1948).

2. H. Nyquist, Phys. Rev. 32., 110, 1928.

Let us remind ourselves, that in Nyquist's derivation two ohmic resistances R are examined, connected by an ideal double transmission line with wave resistance $\sqrt{L/C} = R$ (Fig. 7). Thus, energy exchange between the resistances takes place by means of waves, created in the lines of each of the resistances R and not experiencing reflections in view of complete agreement of the lines with the loads. If at some moment both ends of the line are shortened, then there will be in the line "captured", travelling towards one another, waves with different frequencies ω. The system of standing waves with frequencies lying in the interval $(\omega, \omega + d\omega)$ can be looked upon as a superposition of those natural oscillations of a distinct segment of the line ℓ, whose frequencies are contained in this interval. Applying to these natural oscillations the theorem of equi-partition, Nyquist finds the power which is sent into the line by the emitter in complete harmony with it, namely $\theta\, d\omega/2\pi = kT\, df$. This expression is thus proven for a special form of the line and for the so-called principal waves. The problem, in essence, consists of proving this expression for lines (single channel) of arbitrary form and for any types of waves possible in such a channel and not only for principal waves which, in the general case, could also not be possible (wave guide). In connection with this it is necessary to take into account the dispersion and the difference between phase and group velocities.

Let us note that, given the mentioned proof, we solve the whole problem since the validity of the expression $P_{\omega mn} = \theta A_{mn}/2\pi$, or, in the general case

$$P_{\omega mn} = \frac{2\pi^2 I_{0\omega}}{k^2} A_{mn} \qquad (17.2)$$

for the power, emitted by any (non-harmonized) emitter, can immediately be derived from the requirement of the observation of thermal equilibrium in each spectral interval.

In fact, let emitters be placed in the cross-section of lines A and B (Fig. 8), completely harmonised with the lines at frequency ω on

the wave of some type and number (m,n), i.e., not reflected by this wave. Let us place between them a non-harmonized emitter C. Let the power emitted by it at frequency ω and on the wave (m,n) be $P_{\omega mn}\, d\omega$, and its reflection, absorption and transmission coefficients be, respectively, R_{mn}, A_{mn} and D_{mn} ($R_{mn} + A_{mn} + D_{mn} = 1$). Let us examine the space between the non-harmonized emitter C and B. From B we have, according to assumption, the energy flow $\frac{2\pi^2 I_{0\omega}}{k^2}\, d\omega$. The energy flow from the non-harmonized emitter C toward B is composed of the power $P_{\omega mn}\, d\omega$ emitted by C and also of the radiation from B reflected by C, i.e., $R_{mn} \cdot \frac{2\pi^2 I_{0\omega}}{k^2}\, d\omega$, and, finally, of the emission from A transmitted through C, i.e., $D_{mn} \cdot \frac{2\pi^2 I_{0\omega}}{k^2}\, d\omega$. At equilibrium, the opposite flows must be equal

$$\frac{2\pi^2 I_{0\omega}}{k^2}\, d\omega = \left\{ P_{\omega mn} + \frac{2\pi^2 I_{0\omega}}{k^2} (R_{mn} + D_{mn}) \right\} d\omega$$

from which (17.2) follows. Thus, the problem is in fact reduced to the general proof that the harmonized emitter sends into the line at frequency ω and on the wave (m,n) the power

$$\frac{2\pi^2 I_{0\omega}}{k^2}\, d\omega . \tag{17.3}$$

The phase and group velocities of travelling waves $e^{i(\omega t - \beta z)}$ in any ideal line (waveguide or line with return transport) are, respectively,

$$v = \frac{\omega}{\beta}, \quad w = \frac{c^2}{v} = \frac{c^2 \beta}{\omega}, \tag{17.4}$$

where $\beta = \sqrt{k^2 - \varkappa^2}$, and $\varkappa$ is the natural value of the double boundary problem (of the membrane whose form coincides with the form of the cross-section of the transfer line and which has either a free or fixed contour). We again assume, that in the cross-sections of the line A and B (Fig. 8),

separated from each other by a sufficiently large distance ℓ, are placed emitters, completely harmonized with the line at a frequency ω on the wave with the natural value $\varkappa_{mn}$ (abbreviated: $\varkappa_{mn}$-wave).

At some moment we place at sections A and B ideally reflecting screens and the resulting union of opposed travelling waves of different types and numbers we regard as superposition of natural oscillations of the created volume resonator. The natural frequencies of the resonator are distinguished by the requirement that an integral number of half waves corresponding to these frequencies be contained between A and B, i.e., by the condition

$$\beta = \sqrt{k^2 - \varkappa_{mn}^2} = \frac{s\pi}{\ell} \quad (s = 1,2,3,\ldots) \tag{17.5}$$

whence

$$\omega_{mns} = c\sqrt{\left(\frac{s\pi}{\ell}\right)^2 + \varkappa_{mn}^2}\,. \tag{17.6}$$

On the plane with coordinates $\beta = \pi s/\ell$ and $\varkappa$, an imagined point in the first quadrant corresponds to each natural frequency. In the direction of the axis of abscissae these points are distributed with a constant density $\frac{ds}{d\beta} = \frac{\ell}{\pi}$. Concerning their abundance in the direction of the axis of ordinates, -- it is easy to determine it for sufficiently large values $\varkappa$ from the asymptotic behavior of the natural values of the "membrane". Namely, the number of these natural values $\varkappa_{mn}$, not exceeding $\varkappa$, approaches with increasing $\varkappa$ $S\varkappa^2/4\pi$, where S is the area of the "membrane", i.e., the area of the cross-section of the transfer line.[1] If we take into account both forms of the boundary conditions for the contour of the "membrane", i.e., take into account both E and H waves, then the mentioned quantity must be doubled. Thus, in the interval $d\varkappa$ for large $\varkappa$ are contained $\frac{S\varkappa}{\pi}d\varkappa$ natural values,[2] and

1. R. Courant and D. Hilbert, Methods of Mathematical Physics. Vol. I, Chapt. VI, section 4 (M. - L., 1951).

2. It is not difficult to understand that the proportionality of the

heir number in the element $d\varkappa d\beta$ is

$$dZ = \frac{S}{\pi}\varkappa d\varkappa ds . \tag{17.7}$$

According to (17.6), to a given value ω corresponds a quarter circle of radius ω/c on the plane $(\beta,\varkappa)$ lying in the first quadrant (Fig. 9).

At the moment we are interested only in waves of the $\varkappa_{mn}$-type with frequencies in the interval $(\omega, \omega + d\omega)$. The number of natural oscillations for fixed $\varkappa = \varkappa_{mn}$, for which the frequencies lie in the interval $(\omega, \omega + d\omega)$, as is clear from Fig. 9, is the number of imagined points in the segment $d\beta$. This number is readily obtained by differentiating (17.5) at constant $\varkappa_{mn}$:

$$ds = \frac{\ell}{\pi}\frac{kdk}{\sqrt{k^2 - \varkappa_{mn}^2}} = \frac{\ell\omega d\omega}{\pi c^2\beta} .$$

Introducing here the group velocity w from (17.4) we obtain

$$ds = \frac{\ell d\omega}{\pi w} . \tag{17.8}$$

According to the theory of energy distribution, to each of these natural oscillations corresponds the energy $\varepsilon(\omega,T) = \frac{4\pi^3}{k^2} I_{0\omega}$, so that their total energy is

$$\varepsilon(\omega,T)ds = \frac{4\pi^2\ell I_{0\omega}}{k^2 w} d\omega .$$

But in the presence of active harmonized emitters, the reflection of $\varkappa_{mn}$-waves is absent. For this reason, the expression obtained is the energy, sent into the line by both emitters during the group traversal time of $\varkappa_{mn}$-waves through the distance ℓ, i.e., during the time $\tau = \ell/w$.

density of natural values $\varkappa_{mn}$ along the coordinate $\varkappa$ to $\varkappa$ is a consequence of their uniform distribution according to m and n for large m and n.

dimensions of the line be necessarily very large compared to the wavelength, nevertheless it assumes the fulfillment of condition (17.11), i.e., it has an asymptotic character. The possibility of a classical derivation of (17.1) is related to this fact as well as to the fact that the line contains travelling waves at all [only to these does (17.1) refer.]

Besides the theorem concerning energy distribution, we utilized in this derivation only the estimate of the number of natural (transverse) oscillations of a homogeneous continuum in the frequency interval (ω, $\omega + d\omega$). Hence it is clear that expression (17.1) must be valid not only for feeders but also for any uniform coupling of mutually orthogonal waves. It is because of this that we arrived at the same result (17.1) in the problems of cylinder and sphere radiation, in which concern was with radially propagating waves in cylindrical and spherical coordinates, respectively.[1] Although (17.1) thus embraces a rather larger class of problems, than those of radiation propagation in wave guides, nevertheless, taking into account the mentioned "uniformity" of this result, it may make sense to call it the wave guide form of Kirchhoff's law.

It should be noted that it is precisely the wave guides who make it possible to distinguish waves of individual types and numbers, due to which the individual terms in (17.1) become of direct interest. Since the condition $\hbar\omega \ll \theta$ is satisfied for corresponding frequencies, (17.2) becomes

$$P_{\omega\, mn} = \frac{\theta}{2\pi} A_{mn} . \qquad (17.12)$$

Thus, to calculate the thermal power $P_{\omega\, mn}$ it is necessary to know only the temperature of the emitter and its absorption coefficient for the waves of interest, i.e., the quantity, which, for the case of non-transparent radiation ($D_{mn} = 0$), can be directly measured by the corres-

1. We could have derived (17.1) by the classical method, by examining not a segment of the transfer line, but the space between the surface of an emitting cylinder and a coaxial cylindrical mirror or between a sphere and a concentric spherical mirror.

ponding coefficient of standing waves (CSW).

If the emitting arrangement is partially transparent, then the experimental determination of A_{mn} requires measuring both the reflection coefficient R_{mn} (by CSW) and the transmission coefficient D_{mn}.

Naturally, by absorption coefficient A_{mn} of a given arrangement one must understand the quantity which characterizes the whole arrangement and not only its absorbing elements. If, for instance, we are talking about a semi-transparent thin plate, placed into a wave guide at some distance ℓ from an ideally reflecting piston, then the absorption coefficient A_{mn} of such a system will be a periodic function of ℓ (with the period $\Lambda_{mn}/2$). With a decrease in the transparency of the plate (let us say, by increasing its thickness) the dependence of A_{mn} on ℓ will become weaker and in the limit, for an opaque plate, A_{mn} will become equal to the plate's own absorption coefficient.

The wave guide form of Kirchhoff's law leaves aside, of course, everything that is concerned with the total energy density (to which the quasi-stationary thermal field makes its contribution), as well as with all interference phenomena in systems whose extension in the direction of propagation is comparable to the wavelength. But the evaluation of the power of thermal radiation of any source is considerably simplified and reduces to, in accordance with (17.1) and (17.2), the evaluation of energy absorption coefficients of a given source for those mutually orthogonal waves which form the entire system of waves for the problem under study. The application of an electrodynamic evaluation, utilizing the lateral fluctuating field K or $\mathcal{K}$, becomes for the given problem superfluous since this evaluation embraces not only that which is of interest to us -- the establishment of the form of the coefficients A_{mn} -- but also the proof (17.1) in each concrete case.[1]

1. Radiation of a sphere or of a cylinder is the same in all radial directions. But for a disk, for instance, it will have a definite diagram of directions (indicatrice) and the clarification of its form for various ratios of disk radius to wavelength can in itself be of interest. In such cases, when we are not talking about the total energy flow, but about its angular distribution, the advantages

For well conducting bodies the calculation of the coefficients A_{mn} is considerably simplified due to the possibility of utilizing the approximated boundary conditions (11.6).

As an example, let us examine the thermal radiation of the walls of a feeder (wave guide or coaxial line) assuming the conductivity of the walls to be sufficiently high. In this case it is possible, as is known, to consider the waves in the real line to differ from the natural E and H waves of the line with ideally conducting walls only by the exponential attenuation multipliers. Let γ_{mn} and γ'_{mn} be the energy extinction coefficients of the waves E_{mn} and H_{mn}. The absorption coefficient, let us say of the E_{mn} waves, in the line segment from $z = -\ell$ to $z = 0$, will be, evidently

$$A_{mn} = 1 - D_{mn} = 1 - e^{-\gamma_{mn}\ell} \quad (R_{mn} = 0).$$

In accordance with (17.12), the thermal power, sent on the wave E_{mn} by the mentioned line segment through the section $z = 0$, is

$$P_{\omega\, mn} = \frac{\theta}{2\pi}(1 - e^{-\gamma_{mn}\ell}). \tag{17.13}$$

The total emissive power at frequency ω is

$$P_{\omega} = \frac{\theta}{2\pi}\sum_{m,n}{}' \left(1 - e^{-\gamma_{mn}\ell} + 1 - e^{-\gamma'_{mn}\ell}\right) \tag{17.14}$$

where summation extends over all subcritical waves.

If $\ell \to \infty$, then we obtain the emissive power of the walls of the semi-infinite line

$$P_{\omega} = \frac{\theta}{2\pi} N \tag{17.15}$$

where N is the total number of E and H waves, which are subcritical at

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of the electrodynamic fluctuation calculation, relying on regular methodism of boundary problems, again comes into the foreground.

frequency ω. As ω increases -- in the region of geometrical optics -- formula (17.15) must, evidently, give the radiation from a "black" aperture, having the area S of the cross-section of the line. Acccording to the Rayleigh-Jeans law, the intensity in the case of a "black" aperture is $I_{0\omega} = \frac{\theta k^2}{4\pi^3}$, and the emissive power of the aperture therefore is

$$P_\omega = SI_{0\omega} \int_{\theta < \pi/2} \cos\theta \, d\Omega = \frac{\theta k^2 S}{4\pi^3} 2\pi \int_0^{\pi/2} \cos\theta \sin\theta \, d\theta = \frac{\theta S}{\lambda^2}. \quad (17.16)$$

By calculating the asymptotic (as $\omega \to \infty$) value N in (17.15) for any concrete shape of the line section, it is possible to convince oneself that (17.15) does in fact become (17.16), although this will be shown below in a more general way.

In the case of a coaxial line the principal wave E_{00}, too, enters the E wave. With the help of the boundary conditions (11.6) it is not difficult to establish that, for a round coaxial line, whose central wire alone has a finite conductivity, the extinction coefficient of the principal wave is

$$\gamma_{00} = \delta/2a \ln \frac{b}{a}$$

where $\delta = k\mu d$ (d is the thickness of the skin-layer), and a and b are the radii of the central wire and the external cylinder. As is usual for coaxial lines, let the critical frequencies of all waves of the waveguide type be greater than ω and, consequently, only the principal wave is propagated. Further, let only the segment of the central wire of the length ℓ, have a finite conductivity, and let the temperature be distributed along this segment according to the law $\theta(z)$. Since, according to (17.13), the elementary "belt" of width dz emits the power

$$dP_\omega = \frac{\theta}{2\pi} \gamma_{00} \, dz$$

the total power from segment ℓ will be

$$P_\omega = \frac{1}{2\pi}\int_0^{\ell}\theta(z)\,\gamma_{00}\,dz$$

or -- if we consider γ_{00} to be independent of θ:

$$P_\omega = \frac{\gamma_{00}}{2\pi}\int_0^{\ell}\theta(z)dz = \frac{\ell\delta\overline{\theta}}{4\pi a\ln(b/a)} \qquad (17.17)$$

This formula is useful, of course, also in the case when all surfaces have a finite conductivity, but the temperature of the segment ℓ of the central wire is considerably higher than that of other surfaces, so that their emission can be neglected.

In conclusion, let us note the following. In the derivation of (17.3) we did not utilize the asymptotic behavior of $\varkappa_{mn}$ since the number of natural values (17.5) was determined for a fixed $\varkappa$. However, since this $\varkappa$ could be arbitrary, the result is valid for all m and n, starting with the smallest.

Taking into account the asymptotic behavior of $\varkappa_{mn}$ and, correspondingly, of the natural frequencies (17.6), i.e., going over to the approximation of geometrical optics (when the dimensions of the line section are also large compared to the wavelength), one can very readily obtain some results which had been previously found either indirectly or by means of direct evaluation for concrete special cases. For this, it is convenient to introduce polar coordinates (fig. 9) on the plane $(\beta, \varkappa)$.

$$\beta = \frac{\omega}{c}\cos\varphi, \qquad \varkappa = \frac{\omega}{c}\sin\varphi$$

In these coordinates, (17.7) takes the form

$$dz = \frac{s\ell\omega^2\sin\varphi}{\pi^2c^3}\,d\varphi\,d\omega. \qquad (17.18)$$

Let us evaluate the power in the interval $d\omega$, which is emitted into the line by the radiator, harmonized with the line on waves of all types.

For natural frequencies, lying in the surface elements $\omega\, d\omega d\varphi$, the group velocity is $w = \frac{c^2 \beta}{\omega} = c \cos\varphi$ and the group time of traversal τ equals

$$\tau = \frac{\ell}{c \cos\varphi}. \tag{17.19}$$

During this time, two harmonized radiators send into the volume $S\ell$ the energy θdZ, i.e., each one of them gives the energy $\theta dZ/2\tau$ in one second. Substituting here the expressions (17.18) and (17.19), we obtain

$$\frac{\theta\, dZ}{2\tau} = \frac{\theta S \omega^2 d\omega}{2\pi^2 c^2} \sin\varphi \cos\varphi\, d\varphi .$$

In order to take into account in the interval $d\omega$ waves of all types, this expression must be integrated with respect to φ from 0 to $\pi/2$. We obtain thus the formula (17.16)

$$P_{\omega} d\omega = \frac{\theta S \omega^2}{4\pi^2 c^2} d\omega = \frac{\theta S}{\lambda^2} d\omega .$$

To find the energy of equilibrium radiation, contained in the volume $S\ell$, one must evidently multiply (17.18) by θ and integrate with respect to φ from 0 to $\pi/2$. This gives formula (16.13)

$$U_{\omega} d\omega = \theta \frac{S\ell\omega^2}{\pi^2 c^3} d\omega = S\ell \cdot u_{0\omega} d\omega .$$

CHAPTER IV. EQUILIBRIUM RADIATION IN AN ANISOTROPIC MEDIUM[1]

Section 18. Formulation of the Problem

In this chapter we give ourselves the problem -- to find an expression for the energy intensity and density of equilibrium radiation in a transparent anisotropic medium, i.e., to find an expression which is a generalization of the laws

$$I_\omega = I_{0\omega} n^2, \quad u_\omega = u_{0\omega} n^3, \tag{18.1}$$

referring to a transparent isotropic medium. The second of these laws assumes absence of scattering. Thus, we are speaking of asymptotic relations, which characterize the energy field of travelling (propagating) waves in a non-absorbing medium. This immediately adduces the possibility of obtaining these relations by methods of the classical theory of radiation.

The assumption concerning the transparency of the anisotropic medium permits us to look for the solution of the given problem by means of the generalization of the usual derivation of (18.1) from reflection and refraction of plane waves on the boundary of the two media. In the case of interest to us, one of the media should be taken as the anisotropic medium to be studied and the other medium could be considered to be isotropic or a vacuum.

On the other hand, the fact that we are interested in asymptotic laws for the wave field permits us to apply another classical method based on the asymptoticity of the natural magnitudes. Having determined in some volume V the asymptotic (as $\omega \to \infty$) value of the number dZ of natural field oscillations, whose frequencies lie in the spectral interval $(\omega, \omega + d\omega)$ and by multiplying then $\frac{dZ}{V}$ by the average energy $\varepsilon(\omega, T)$ of the oscillator, we can obtain the general expression for the equilibrium density of energy u_ω. Using then the optical laws of anisotropic media, we can then obtain the general formula for the equilibrium

1. Under anisotropic we also include magnetoactive media.

intensity I_ω from u_ω.

The first of the mentioned methods requires only a few calculations less than the electrodynamic method which utilizes the concept of the lateral fluctuating field. The second method brings us to our goal by a much shorter path, since the asymptotic expression in dZ begins to operate immediately. Nevertheless, we shall construct the exposition in the same manner as we did in the previous chapter. We shall first examine that solution of the problem which is obtainable by applying the electrodynamic fluctuation theory (sections 19-22), then we shall expound the classical method, using the asymptotic value of dZ (section 23), and then we shall discuss the final results (section 24).

Within the framework of electrodynamic fluctuation theory too, different paths can be followed towards the establishment of laws which generalize (18.1) for the case of a transparent anisotropic medium. We shall indicate here two methods, of which the first presents only in principle an interest for the problem of equilibrium radiation within an anisotropic medium.

Discussion now concerns itself with the generalization of the correlation function of components of the lateral field $\vec{K}$ to the case of an anisotropic absorbing medium in order to obtain an expression for I_ω and u_ω within such a medium. As had been clarified in section 8, in order to have these expressions finite, it is necessary to examine the correlation with a radius different from zero. By separating out those terms in I_ω and u_ω which are independent of the correlation radius and by passing to the limiting case of no attenuation, we would obtain the sought expressions for I_ω and u_ω in a transparent absorbing medium.

One can see that this path is extremely complicated from the evaluation point of view; but the basic difficulty is, of course, due to renunciation of the δ-correlation. The generalization of the correlation function for the case of an anisotropic medium by preservation of the δ-correlation appears to be relatively simple. It is evident that one must go from the function (3.5)

$$F_{\alpha\beta}(r) = C\delta_{\alpha\beta}\sigma(\vec{r}) \qquad (18.2)$$

to the more general function

$$F_{\alpha\beta}(r) = C_{\alpha\beta}\sigma(\vec{r}), \qquad (18.3)$$

where $C_{\alpha\beta}$ is the constant symmetrical tensor. For the determination of the poles of its components it is necessary (in a way similar as that in section 6 for the determination of the constant C) to refer to radiation of the anisotropic medium under study into the exterior, for instance to investigate the emission of a half-space filled with such a medium. But, as has been said, this would not advance as in the solution of the problem formulated above since it is necessary to examine a correlation radius different from zero for the determination of I_{α} and u_{α} within an absorbing medium. The generalization (18.2) can no longer be reduced to the replacement of $C\delta_{\alpha\beta}$ by $C_{\alpha\beta}$, but requires a significant complication in the form of the function $\vec{r}$ itself, since the correlation radius can be different for different directions.

It should be noted that although the determination of $C_{\alpha\beta}$ in (18.3) does not permit us to find I_{α} and u_{α} within an absorbing anisotropic medium, it nevertheless can be of interest for problems concerning the radiation of such a medium into outer space.

The second method, on which we shall base ourselves, is considerably simpler. It does not require the generalization of the correlation function and is a natural consequence of previously obtained results.

From the start, we consider the medium of interest to us to be transparent; and we shall take isotropic bodies, in which the fluctuations have already been studied by us, as emission sources for equilibrium radiation in this medium. Since we are talking about equilibrium radiation, the result will not depend on parameters characterizing the substances of the emitter sources. We can therefore take a body with such a large value for $|\varepsilon|$, that the utilization of boundary conditions (11.7) will be valid. Thereby we shall be able to examine only the field of interest to us in the anisotropic medium itself.

Further, among the systems capable of giving equilibrium radiation it is expedient to choose the geometrically most simple one, namely, the plane waveguide we had already used in section 7. As was done there, it is sufficient to consider only one of the bounding planes of the system as the emitter and to take an ideal mirror for the other plane. Since the desired laws have, as in (18.1), an asymptotic character, we must consider the distance ℓ between the planes to be sufficiently large and we must average over ℓ the energy quantities to be obtained, which results in a smearing of the interferential effects (see section 7).

Finally, we dispose of the possibility of orienting, in an arbitrary fashion, the axes of the anisotropic medium with respect to the waveguide and, consequently, we can arrange the direction of these axes in such a way as to simplify the evaluations to the greatest extent possible.

And thus, the problem is formulated in the following way. The space between the emitter plane $z = 0$, at which the boundary conditions (11.7) are satisfied[1]

$$\sqrt{\varepsilon}\, E_1 + H_2 = -\sqrt{\varepsilon}\, \mathcal{X}_1 \qquad (18.4)$$

$$\sqrt{\varepsilon}\, E_2 - H_1 = -\sqrt{\varepsilon}\, \mathcal{X}_2$$

and the ideal mirror which is located at the plane $z = \ell$ and at which

$$E_1 = 0, \qquad E_2 = 0, \qquad (18.5)$$

is filled with a homogeneous transparent anisotropic medium (Fig. 2). The magnetic permeability of this medium will be taken equal to unity. Its dielectric permeability is expressed, in the general case, by the Hermite tensor $\tilde{\varepsilon}$ with components

$$\varepsilon_{ik} = \varepsilon^{*}_{ki} \,. \qquad (18.6)$$

1. We shall convince ourselves later that the boundary conditions cannot be taken in th[illegible] form (11.8). The material of the "noisy" wall is assumed [illegible]tic, $\mu = 1$.

The field equations in the medium have the form

$$\text{curl}\ \vec{E} = -ik\vec{H} \qquad (18.7)$$
$$\text{curl}\ \vec{H} = ik\varepsilon\vec{E}$$

i.e., the components of the vector of electric inductance $\vec{D} = \varepsilon\vec{E}$ are equal to

$$D_i = \sum_{k=1}^{3} \varepsilon_{ik} E_k \qquad (i = 1,2,3).$$

Our initial aim will be to find the average value of components $S_{\omega 3}$ of the energy flow density of waves going from the "noisy" surface to the mirror. Speaking of the average value, we have in mind an averaging in two senses. First, an overall averaging by means of the correlation function (12.13) of the components of $\mathcal{K}$ or of the correlation function (12.11) of the component Fourier-conjugate to g

$$\overline{g_\alpha g_\beta^*} = \frac{e\delta_{\alpha\beta}}{(2\pi)^2}\delta(p_1 - p_1')\delta(p_2 - p_2'). \qquad (18.8)$$

Second, averaging over the waveguide width ℓ, which, beforehand, is considered to be large compared to the maximum wavelength in the medium. Due to this assumption and the averaging over ℓ, we pass from the general expression for $S_{\omega 3}$ to the asymptotic approximation which is the only thing that we are interested in in the formulated problem. Subsequently, we find the expression for the intensity I_ω from $S_{\omega 3}$.

In the next section, we shall establish some general expressions, independent of any assumptions concerning the nature of anisotropy, except for the basic assumption (18.6) and the as yet unused possibility of choosing the direction for the axes of symmetry of the medium.

Section 19. General Relations

We shall look for the particular solution of equations (18.7) in

the form of a plane wave[1]

$$\vec{E} = \vec{u}\, e^{-i\vec{p}\vec{r}}, \qquad \vec{H} = \frac{1}{k}\,[\vec{p}, \vec{u}]\, e^{-i\vec{p}\vec{r}}. \tag{19.1}$$

The first equation (18.7) in this solution has already been utilized and substitution into the second equation gives for $\vec{u}$ a system of three homogeneous algebraic equations

$$\sum_{k=1}^{3} (p^2 \delta_{ik} - p_i p_k - \alpha_{ik})\, u_k = 0, \tag{19.2}$$

where the following notation has been used

$$p^2 = p_1^2 + p_2^2 + p_3^2, \qquad \alpha_{ik} = k^2 \varepsilon_{ik}. \tag{19.3}$$

From (19.2) follows the condition of transversality for the electric inductance vector

$$\sum_{i,k=1}^{3} \alpha_{ik} u_k p_i = 0. \tag{19.4}$$

The condition for the existance of a non-trivial solution for equations (19.2) is that the determinant be zero for the system, which upon taking into account (18.6) has the form

$$f(\omega, \vec{p}) \equiv \begin{vmatrix} p_2^2 + p_3^2 - \alpha_{11} & -p_1p_2 - \alpha_{12} & -p_1p_3 - \alpha_{31}^* \\ -p_1p_2 - \alpha_{12}^* & p_1^2 + p_3^2 - \alpha_{22} & -p_2p_3 - \alpha_{23} \\ -p_1p_3 - \alpha_{31} & -p_2p_3 - \alpha_{23}^* & p_1^2 + p_2^2 - \alpha_{33} \end{vmatrix} = 0. \tag{19.5}$$

1. In order for $p_3 > 0$ to correspond to the wave propagation in the positive direction of the axis z, we shall now take, contrary to section 8, the minus sign in the exponent. A change in sign of the components p_1 and p_2, evidently, does not affect the form of the function (18.8).

This is the so-called scattering equation which relates the frequency ω, on which α_{ik} depends, to the components of the waveguide vector p_1, p_2, p_3. With respect to p_3, this equation is of the fourth degree with real coefficients.

If the main axes (or axes of symmetry) of the medium are made to coincide with the coordinate axes, then the tensor α_{ik} for uni- and di-axial crystals becomes diagonal, and for a magneto-active medium in a magnetic field directed along the z-axis only α_{12} among the non-diagonal components is different from zero. For all these cases equation (19.5) becomes bi-quadratic

$$ap_3^4 + bp_3^2 + c = 0 \tag{19.6}$$

and, consequently, its roots appear in pairs not only with respect to the conjugates but with respect to sign. If we denote the two roots corresponding to waves which are propagating (or are extinguished) in the direction from the "noisy" wall to the mirror by p_3 and q_3, then the other two roots will be $-p_3$ and $-q_3$. Let us therefore introduce four wave vectors $\vec{p}$, $\vec{q}$, $\vec{s}$ and $\vec{t}$ such that

$$\begin{gathered} p_\alpha = q_\alpha = s_\alpha = t_\alpha \quad \text{for} \quad \alpha = 1,2 \\ s_3 = -p_3, \quad t_3 = -q_3. \end{gathered} \tag{19.7}$$

Let us further denote the amplitudes of the electric field for each of these waves, respectively, by $\vec{u}$, $\vec{v}$, $\vec{U}$ and $\vec{V}$. Then the general solution of equation (18.7) can be written in the form

$$\begin{gathered} \vec{E} = \int_{-\infty}^{+\infty} \left\{ \vec{u}\, e^{-i\vec{p}\vec{r}} + \vec{w}\, e^{-i\vec{q}\vec{r}} + \vec{U}\, e^{-i\vec{s}\vec{r}} + \vec{V}\, e^{-i\vec{t}\vec{r}} \right\} dp_1 dp_2, \\ \vec{H} = \frac{1}{k} \int_{-\infty}^{+\infty} \left\{ [\vec{p},\vec{u}] e^{-i\vec{p}\vec{r}} + [\vec{q},\vec{v}] e^{-i\vec{q}\vec{r}} + [\vec{s},\vec{U}] e^{-i\vec{s}\vec{r}} + [\vec{t},\vec{V}] e^{-i\vec{t}\vec{r}} \right\} dp_1 dp_2 \end{gathered} \tag{19.8}$$

where for each wave vector and its corresponding amplitude equations (19.2)

and the consequent relations (19.4) are satisfied.

The vector for the energy flow density for waves directed toward the mirror will be

$$S_\omega = \frac{c}{4\pi}\left\{\overline{[\vec{E}, \vec{H}^*]} + \overline{[\vec{E}^*, \vec{H}]}\right\} =$$

$$= \frac{c}{4\pi k}\iint_{-\infty}^{+\infty} \overline{[\vec{u}e^{-i\vec{p}\vec{r}} + \vec{v}e^{-i\vec{q}\vec{r}}, \left\{[\vec{p}^*, \vec{u}^*]e^{i\vec{p}^*\vec{r}} + [\vec{q}^*, \vec{v}^*]e^{i\vec{q}^*\vec{r}}\right\}]} \cdot$$

$$\cdot\, dp_1 dp_2 dp_1' dp_2' + \text{complex conjugate}$$

We are interested only in component $S_{\omega 3}$ for which, taking (19.7) into account, we get

$$S_{\omega 3} = \frac{c}{4\pi k}\iint_{-\infty}^{+\infty}\left\{Pe^{-i(p_3-p_3^*)z} + Qe^{-i(q_3-q_3^*)z} + \right.$$
$$\left. + Re^{-i(p_3-q_3^*)z} + R^*e^{-i(q_3-p_3^*)z}\right\} dp_1 dp_2 dp_1' dp_2' , \tag{19.9}$$

where

$$P = (p_3 + p_3^*)(\overline{u_1u_1^*} + \overline{u_2u_2^*}) - p_1(\overline{u_1u_3^*} + \overline{u_1^*u_3}) - p_2(\overline{u_2u_3^*} + \overline{u_2^*u_3}) ,$$
$$Q = (q_3 + q_3^*)(\overline{v_1v_1^*} + \overline{v_2v_2^*}) - p_1(\overline{v_1v_3^*} + \overline{v_1^*v_3}) - p_2(\overline{v_2v_3^*} + \overline{v_2^*v_3}) , \tag{19.10}$$
$$R = (p_3 + q_3^*)(\overline{u_1v_1^*} + \overline{u_2v_2^*}) - p_1(\overline{u_1v_3^*} + \overline{u_3v_1^*}) - p_2(\overline{u_2v_3^*} + \overline{u_3v_2^*}) .$$

The mean values entering (19.10) are proportional to $\delta(p_1-p_1')\,\delta(p_2-p_2')$, so that we can consider that in the multipliers of these δ-functions $p_1' = p_1$ and $p_2' = p_2$.

The roots p_3 and q_3 correspond to waves with two independent polarizations -- linearly orthogonal between themselves in the case of simple anisotropy and elliptically in the case of a magneto-active medium. Thus,

these waves are not coherent and in view of this the expression R, in which they are combined, must always become zero. The direct evaluation carried out below gives precisely this result.

Let us now return to the boundary conditions. In accordance with (19.8), we utilize the expansion of the lateral field $\mathcal{H}$ on the plane $z = 0$ into a double Fourier integral

$$\mathcal{H} = \int_{-\infty}^{\infty} \vec{\mathcal{G}}(p_1, p_2) e^{-i(p_1 x + p_2 y)} dp_1 \, dp_2 .$$

Substituting this expansion and (19.8) into the boundary conditions (18.4) and (18.5) and taking again into account (19.7), we find

$$- p_3(u_1 - U_1) - q_3(v_1 - V_1) - k\sqrt{\varepsilon}(u_1 + U_1 + v_1 + V_1) +$$

$$+ p_1(u_3 + U_3 + v_3 + V_3) = k\sqrt{\varepsilon}\,\mathcal{G}_1 ,$$

$$- p_3(u_2 - U_2) - q_3(v_2 - V_2) - k\sqrt{\varepsilon}(u_2 + U_2 + v_2 + V_2) +$$

$$+ p_2(u_3 + U_3 + v_3 + V_3) = k\sqrt{\varepsilon}\,\mathcal{G}_2 , \qquad (19.11)$$

$$u_1\sigma + \frac{U_1}{\sigma} + v_1\rho + \frac{V_1}{\rho} = 0, \qquad u_2\sigma + \frac{U_2}{\sigma} + v_2\rho + \frac{V_2}{\rho} = 0 ,$$

where the following notation is introduced

$$\sigma = e^{ip_3 \ell}, \qquad \rho = e^{iq_3 \ell} . \qquad (19.12)$$

Equation (19.2), written for each of the four roots p_3, q_3, s_3 and t_3, leaves exactly four independent components (out of twelve components $\vec{u}$, $\vec{v}$, $\vec{U}$ and $\vec{V}$), which are determined by $\mathcal{G}_1$ and $\mathcal{G}_2$ with the help of the four equations (19.11).

The evaluation of energy flow (19.9) could be carried out for the general case when all the diagonal elements of the tensor $\widetilde{\alpha} - k^2 \widetilde{\varepsilon}$ are

distinct and, besides, $\alpha_{12} = 0$. However, the evaluation then becomes exceedingly cumbersome. It is therefore expedient to examine particular forms of anisotropy separately. In the next section we shall present the complete derivation only for an uniaxial crystal, but for the magnetoactive medium and for bi-axial crystals we shall adduce only the original relations and the final result.

Section 20. Asymptotic Expression for Equilibrium Energy Flow in an Anisotropic Medium

In the case of an uni-axial crystal, if its axis be directed along the z-axis, the tensor $\tilde{\alpha}$ assumes the form

$$\tilde{\alpha} = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_1 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}, \quad \alpha_1 = k^2 \varepsilon_1, \quad \alpha_3 = k^2 \varepsilon_3 . \tag{20.1}$$

Equation (19.6) decomposes into the multipliers

$$[\alpha_3 p_3^2 - \alpha_1(\alpha_3 - p^2)][p_3^2 - (\alpha_1 - p^2)] = 0$$

where

$$p^2 = p_1^2 + p_2^2 . \tag{20.2}$$

Thus

$$p_3 = \sqrt{\alpha_1 - p^2}, \quad q_3 = \sqrt{\frac{\alpha_1}{\alpha_3}}\sqrt{\alpha_3 - p^2} . \tag{20.3}$$

The roots p_3 and $s_3 = -p_3$ correspond to ordinary waves and are degenerate, i.e., they render not only the determinant (19.5) zero, but also its minors of the second order. Consequently, for these three roots only one of the three equations (19.2) remains independent, for

example the first. Using the expression for p_3, it is not difficult to convince oneself that this equation reduces to the condition of transversality for the electric potential

$$(p_2^2 + p_3^2 - \alpha_1)u_1 - p_1p_2u_2 - p_1p_3u_3 = -p_1(p_1u_1 + p_2u_2 + p_3u_3) = 0 .$$

From here and from the condition of transversality for electric induction (19.4), which now has the form

$$\alpha_1(p_1u_1 + p_2u_2) + \alpha_3p_3u_3 = 0$$

it follows that

$$u_2 = -\frac{p_1}{p_2}u_1, \quad u_3 = 0 . \tag{20.4}$$

Analogous expressions for the ordinary reflected wave, i.e., for $s_3 = -p_3$ and $\vec{U}$, give a similar result

$$U_2 = -\frac{p_1}{p_2}U_1 , \quad U_3 = 0 . \tag{20.5}$$

Let us now examine the roots q_3 and $t_3 = -q_3$ which correspond to the extraordinary waves. These roots do not render the minors of the second order zero, i.e., they leave two independent equations (19.2) which permit us to find the relations of components $\vec{v}$ for q_3 and of $\vec{V}$ for t_3. We shall express all components in terms of v_1 and V_1 :

$$v_2 = \frac{p_2}{p_1}v_1 , \quad v_3 = \sqrt{\frac{\alpha_1}{\alpha_3}}\frac{p^2}{p_1\sqrt{\alpha_3 - p^2}}v_1 \tag{20.6}$$

$$V_2 = \frac{p_2}{p_1}V_1 , \quad V_3 = -\sqrt{\frac{\alpha_1}{\alpha_3}}\frac{p^2}{p_1\sqrt{\alpha_3 - p^2}}V_1 . \tag{20.7}$$

In the end all components are expressed in terms of four u_1, v_1, U_1 and V_1.

It is easy to convince oneself that the orthogonality of polarization in ordinary and extraordinary waves and the rendering of the coefficient R in (19.10) zero follow from the expressions found between the components. Concerning the coefficients P and Q: by virtue of (20.4) and (20.6) they are considerably simplified and the expression (19.9) for $S_{\omega 3}$ takes the following form:

$$S_{\omega 3} = \frac{c}{4\pi k} \iint_{-\infty}^{+\infty} \left\{ \frac{p_3 + p_3^*}{p_2^2} \overline{u_1 u_1^*} \, e^{-i(p_3 - p_3^*)z} + \right.$$

$$\left. + \frac{\alpha_1}{p_1^2} \left(\frac{1}{q_3} + \frac{1}{q_3^*} \right) \overline{v_1 v_1^*} \, e^{-i(q_3 - q_3^*)z} \right\} p^2 \, dp_1 \, dp_2 \, dp_1' \, dp_2'$$

where the first and second terms give the energy flow of ordinary and extraordinary waves, respectively. Since p_3 and q_3 can be either real (travelling waves) or purely imaginary (standing, non-uniform waves) for which $p_3 + p_3^* = 0$ and $\frac{1}{q_3} + \frac{1}{q_3^*} = 0$, integrals will remain in $S_{\omega 3}$ only over the domains of real p_3 and real q_3. Let us denote these domains, respectively, by 0 (ordinary waves, $p^2 < \alpha_1$) and e (extraordinary waves, $p^2 < \alpha_3$).

Then

$$S_{\omega 3} = \frac{c}{2\pi k} \left\{ \int_0 \frac{p_3 p^2}{p_2^2} \, dp_1 \, dp_2 \int_{-\infty}^{+\infty} \overline{u_1 u_1^*} \, dp_1' \, dp_2' + \right.$$

$$\left. + \int_e \frac{\alpha_1 p^2}{p_1^2 q_3} \, dp_1 \, dp_2 \int_{-\infty}^{+\infty} \overline{v_1 v_1^*} \, dp_1' \, dp_2' \right\} . \qquad (20.8)$$

To find $S_{\omega 3}$ it is thus necessary for us to evaluate $\overline{u_1 u_1^*}$ and $\overline{v_1 v_1^*}$.

Substitution of (20.3) - (20.7) into the boundary conditions (19.11) transforms the latter to the form

$$[p_3(u_1 - U_1) + k\sqrt{\varepsilon}(u_1 + U_1)] +$$

$$+ \left[\frac{\alpha_1}{q_3}(v_1 - V_1) + k\sqrt{\varepsilon}(v_1 + V_1)\right] = - k\sqrt{\varepsilon}\,\varphi_1$$

$$p_1^2[p_3(u_1 - U_1) - k\sqrt{\varepsilon}(u_1 + U_1)] -$$

$$- p_2^2\left[\frac{\alpha_1}{q_3}(v_1 - V_1) + k\sqrt{\varepsilon}(v_1 + V_1)\right] = k\sqrt{\varepsilon}\,\varphi_2$$

$$U_1 = - \sigma^2 u_1, \quad V_1 = - \rho^2 v_1$$

whence

$$u_1 = \frac{p_2 k\sqrt{\varepsilon}}{p^2} \cdot \frac{p_2\varphi_1 - p_1\varphi_2}{k\sqrt{\varepsilon}(1 - \sigma^2) + p_3(1 + \sigma^2)}$$

$$v_1 = \frac{p_1 k\sqrt{\varepsilon}}{p^2} \cdot \frac{p_1\varphi_1 + p_2\varphi_2}{k\sqrt{\varepsilon}(1 - \rho^2) + \frac{\alpha_1}{q_3}(1 + \rho^2)} .$$

With the help of the correlation function (18.8) we obtain from the above expressions

$$\overline{u_1 u_1^*} = \frac{k^2|\varepsilon|C\, p_2^2}{(2\pi)^2 p^2} \cdot \frac{\delta(p_1 - p_1')\delta(p_2 - p_2')}{|k\sqrt{\varepsilon}(1 - \sigma^2) + p_3(1 + \sigma^2)|^2}$$

$$\overline{v_1 v_1^*} = \frac{k^2|\varepsilon|\mathcal{C} p_1^2}{(2\pi)^2 p^2} \cdot \frac{\delta(p_1 - p_1')\delta(p_2 - p_2')}{\left|k\sqrt{\varepsilon}(1 - \rho^2) + \frac{\alpha_1}{q_3}(1 + \rho^2)\right|^2} .$$

Carrying this into (20.8) and using the value (12.16) of the constant $\mathcal{C}$

$$\mathcal{C} = \frac{8\pi^3(\sqrt{\varepsilon^*} + \sqrt{\varepsilon})}{k^2 c|\varepsilon|} I_{0\omega}$$

we find

$$s_{\omega 3} = \frac{(\sqrt{\varepsilon^*} + \sqrt{\varepsilon})I_{0\omega}}{k}\left\{\int_0 \frac{p_3\, dp_1\, dp_2}{|k\sqrt{\varepsilon}(1 - \sigma^2) + p_3(1 + \sigma^2)|^2} + \right. \tag{20.9}$$

$$\left. + \alpha_1 \int_a \frac{dp_1\, dp_2}{q_3\left|k\sqrt{\varepsilon}(1 - \rho^2) + \frac{\alpha_1}{q_3}(1 + \rho^2)\right|^2}\right\} .$$

It remains for us now to average this expression with respect to $\xi = p_3\ell$ and $\eta = q_3\ell$.

According to (19.12), we have

$$|k\sqrt{\varepsilon}(1 - \sigma^2) + p_3(1 + \sigma^2)|^2 = 4|p_3 \cos\xi - ik\sqrt{\varepsilon}\sin\xi|^2$$

$$|k\sqrt{\varepsilon}(1 - \rho^2) + \frac{\alpha_1}{q_3}(1 + \rho^2)|^2 = 4\left|\frac{\alpha_1}{q_3}\cos\eta - ik\sqrt{\varepsilon}\sin\eta\right|^2$$

and also, according to formula (7.10)

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{p_3\, d\xi}{4|p_3 \cos\xi - ik\sqrt{\varepsilon}\sin\xi|^2} = \frac{1}{2k(\sqrt{\varepsilon^*} + \sqrt{\varepsilon})}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\alpha_1 d\xi}{4q_3\left|\frac{\alpha_1}{q_3}\cos\eta - ik\sqrt{\varepsilon}\sin\eta\right|^2} = \frac{1}{2k(\sqrt{\varepsilon^*}+\sqrt{\varepsilon})} .$$

Here, by the way, it can be seen that the utilization of the limiting form of the boundary conditions would have given in the denominators under the integrals only terms with $\sin\xi$ and $\sin\eta$, i.e., it would not have permitted averaging with respect to ℓ.

Due to averaging with respect to ξ and η we finally obtain

$$S_{\omega 3} = \frac{I_{0\omega}}{2k^2}\left\{\int_0 dp_1\, dp_2 + \int_e dp_1\, dp_2\right\} . \qquad (20.10)$$

We do not evaluate the elementary integrals entering this expression, since we are not interested in the energy flow value $S_{\omega 3}$ but in representation of the flow in a form which would permit the determination of the intensity I_ω (section 21) of equilibrium radiation.

Turning now to the magnetoactive medium, we shall concretely have in mind an ionized gas, i.e., a medium of the ionosphere type, located in the constant magnetic field $\mathcal{H}$. For such a medium, the relation between the induction $\vec{D}$ and the electric field potential $\vec{E}$ is given by the formula[1]

$$\vec{D} = \vec{E} - \frac{1-n_0^2}{1-\vec{\Omega}^2}\left\{\vec{E} - \vec{\Omega}(\vec{\Omega},\vec{E}) - i[\vec{\Omega},\vec{E}]\right\}$$

where

$$1 - n_0^2 = \frac{4\pi N e^2}{m\omega^2}, \qquad \vec{\Omega} = \frac{e\vec{\mathcal{H}}}{mc\omega} \qquad (20.11)$$

N is the electron concentration, e and m are the charge and mass of the

1. See, for instance, V. L. Ginsburg, Theory of Radio-wave Propagation in the Ionosphere, p. 58 (M.-L., 1949).

electron. Thus, n_0 is the index of refraction of the isotropic medium when $\mathcal{H} = 0$, and Ω is the ratio of the gyromagnetic frequency to the field frequency ω. If, along with the gyromagnetic frequency ω_H we introduce the natural frequency ω_0 of the plasma:

$$\omega_0^2 = \frac{4\pi N e^2}{m}, \qquad \omega_H = \frac{e\mathcal{H}}{mc} \tag{20.12}$$

then

$$1 - n_0^2 = \frac{\omega_0^2}{\omega^2}, \qquad \Omega = \frac{\omega_H}{\omega}. \tag{20.13}$$

As in the case of an uni-axial crystal, it is expedient to direct the symmetry axis (i.e., the magnetic field $\mathcal{H}$) along the z-axis. Then the tensor $\widetilde{\alpha} = k^2\widetilde{\varepsilon}$ takes the form

$$\widetilde{\alpha} = \begin{pmatrix} \alpha_1 & -i\beta & 0 \\ i\beta & \alpha_1 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix} \tag{20.14}$$

where

$$\alpha_1 = k^2 \frac{n_0^2 - \Omega^2}{1 - \Omega^2} = k^2 \frac{\omega^2 - \omega_0^2 - \omega_H^2}{\omega^2 - \omega_H^2}$$

$$\alpha_3 = k^2 n_0^2 = k^2 \frac{\omega^2 - \omega_0^2}{\omega^2} \tag{20.13}$$

$$\beta = k^2 \frac{(1 - n_0^2)\Omega}{1 - \Omega^2} = k^2 \frac{\omega_0^2 \omega_H}{(\omega^2 - \omega_H^2)\omega} .$$

Formally, if we do not take into account that α_1, as well as β, vary with change in Ω, we obtain from (20.14) for $\beta = 0$ the tensor $\widetilde{\alpha}$ for

an uni-axial crystal, i.e., we return to the case examined above.

In the biquadratic equation (19.6)

$$ap_3^4 + bp_3^2 + c = 0 \qquad (20.16)$$

for the case when the tensor $\tilde{\alpha}$ has the form (20.14), the coefficients are expressed in the following manner:

$$a = \alpha_3, \quad b = (\alpha_1 + \alpha_3)p^2 - 2\alpha_1\alpha_3,$$

$$c = (p^2 - \alpha_3)(\alpha_1 p^2 - \alpha_1^2 + \beta^2), \qquad (20.17)$$

where the notation $p^2 = p_1^2 + p_2^2$ is used as before. The roots of equation (20.16), which, for $\beta = 0$, become the corresponding roots (20.3) for an uni-axial crystal, are

$$p_3 = \sqrt{\frac{-b + \sqrt{b^2 - 4ac}}{2a}}, \quad q_3 = \sqrt{\frac{-b - \sqrt{b^2 - 4ac}}{2a}}. \qquad (20.18)$$

As before we shall call the waves, corresponding to these roots and polarized, in the general case, along an ellipse, "ordinary" and "extraordinary". Both roots (20.18) are not degenerate and, therefore, for each of them equations (19.2) determine the relations between the components of the electric vector.

The subsequent evaluation is carried out according to the same scheme as was used for an uni-axial crystal but is much more unwieldy. The expression for the energy flow from the "noisy" waveguide wall likewise has the form (20.10) in the asymptotic case (after averaging according to ℓ). The distinction from the uni-axial crystal case consists in the fact that the domains of real values of p_3 and q_3 (the domains O and e) now have a more complex form and vary appreciably with the relation between the parameters (20.13). However, there is no need to examine these changes.

In the case of a bi-axial crystal the tensor $\tilde{\alpha}$ is diagonal, but

all three main values are distinct. The coefficients of the biquadratic equation (20.16) are

$$a = \varepsilon_3, \quad b = p_1^2(\varepsilon_1 + \varepsilon_3) + p_2^2(\varepsilon_2 + \varepsilon_3) - \varepsilon_3(\varepsilon_1 + \varepsilon_2),$$

$$c = (p^2 - \varepsilon_3)(\varepsilon_1 p_1^2 + \varepsilon_2 p_2^2 - \varepsilon_1 \varepsilon_2)$$

and the roots (20.18), which for $\varepsilon_1 = \varepsilon_2$ become the roots (20.3) for an uni-axial crystal, are not both degenerate. Once again, (20.10) is the final formula for energy flow. Here, if $\varepsilon_1 > \varepsilon_2 > \varepsilon_3$, then the domain O is bounded by the ellipse $\varepsilon_1 p_1^2 + \varepsilon_2 p_2^2 = \varepsilon_1 \varepsilon_2$ and the domain e by the circle $p^2 = \varepsilon_3$.

Thus, the expression (20.10) for energy flow is the same in all cases and only the boundaries of the domains O and e, in which p_3 and q_3, respectively, are real, are different. In the next section, after establishing some general relations valid for an anisotropic medium, we shall obtain from (20.10) formulae for equilibrium intensity I_ω and energy density u_ω.

Section 21. Derivation of Formulae for Energy Intensity and Density

Among the very general assumptions concerning the properties of a non-absorbing medium we can introduce the concept of group velocity $\vec{w}$. The direction of energy flow then coincides with the direction of $\vec{w}$ [1] and, generally speaking, is distinct from the direction of the wave normal, i.e., the wave vector $\vec{p}$. In an anisotropic medium this takes place even when the medium is homogeneous and has no scattering, i.e., the tensor $\tilde{\varepsilon}$ does not depend on the frequency ω [2]. For this reason, in energy quantity relations containing the group velocity, it is necessary, in general, to distinguish between group and phase velocities not only with respect

1. S. M. Rytov, JETF 17, 930, 1947.

2. The group velocity in anisotropic media is the same as the beam velocity usually introduced in optics.

to magnitude (this suffices for the case of a homogeneous isotropic medium) but also with respect to direction.

Thus, for instance, the relation between energy intensity and density of equilibrium radiation must not be established by means of formula (5.9)

$$u_{\omega} = \int \frac{I_{\omega} d\Omega}{w}$$

in which the coinciding of the directions of group and phase velocities is assumed ($d\Omega$ solid angle element with axis along the wave normal), but by means of the more general formula

$$u_{\omega} = \int_{\Omega_0} \frac{I_0 d\Omega_0}{w_0} + \int_{\Omega_e} \frac{I_e d\Omega_e}{w_e}, \quad I_{\omega} = I_0 + I_e, \qquad (21.1)$$

where I_0 and I_e are the intensities for the waves of both independent polarizations which are possible in the given medium, w_0 and w_e are the group velocities of these waves, and $d\Omega_0$ and $d\Omega_e$ are the solid angle elements with axes along the group velocities w_0 and w_e. Under Ω_0 and Ω_e we understand the domains of those directions w_0 and w_e which correspond to the propagation of (travelling) plane waves (see below).

Similarly, concerning the determination of intensities, the expression for the density of energy flow through a unit surface with normal N is, in general, given not by formula (5.7)

$$S_{\omega N} = \int_{\theta < \pi/2} I_{\omega} \cos\theta \, d\Omega$$

valid only for an isotropic medium, but by the generalized formula

$$S_{\omega N} = \int_{\Omega_0} I_0 \cos\xi_0 \, d\Omega_0 + \int_{\Omega_e} I_e \cos\xi_e \, d\Omega_e \qquad (21.2)$$

where ξ_0 and ξ_e are the angles, formed by the group velocities w_0 and w_e with the normal N to the surface. Ω_0 and Ω_e in (21.2) once again denote those regions of the directions of w_0 and w_e which correspond to the travelling waves, but not only within the bounds of a positive hemi-sphere, i.e., for acute angles ξ_0 and ξ_e. In this connection it is necessary to emphasize the following.

Up to now, we had been assuming for simplicity that if the direction $\vec{p}$ lies within the limits of a positive hemisphere ($p_N \geqslant 0$, i.e., $0 \leqslant \theta \leqslant \pi/2$), then the energy transfer through the surface in the direction of the normal N is thereby assured. In other words, we implicitly admit that to the indicated direction $\vec{p}$ corresponds the direction of $\vec{w}$, similarly distribu within the limits of the positive hemisphere ($w_N \geqslant 0$). However, a case is possible when this is not so, i.e., to the positive directions of the group velocity can correspond waves whose phase approaches the surface under consideration ($p_N \leqslant 0$).[1] Evidently the determination, given above, of the region of integration for Ω_0 and Ω_e in formula (21.2) is more general and one must keep precisely this determination in mind if the energy flow is represented by means of some other variables and one needs to establish the region of integration in terms of these variables.

The problem now consists of obtaining expressions for the equilibrium intensities I_0 and I_e of waves of the two independent polarizations by transforming the energy flow expression

$$S_{\omega 3} = \frac{I_0 \omega}{2k^2}\left\{ \int_0 dp_1 \; dp_2 + \int_e dp_1 \; dp_2 \right\} \tag{21.3}$$

found above to the form (21.2). In this, we shall limit ourselves to media with axially symmetric anisotropy and shall consider, as had been done in the problem on radiation in a plane waveguide, that the axis of symmetry coincides with the coordinate axis z. Since the following

1. This is the case of the so-called negative group velocities. In another connection, the possibility of such phenomena was pointed out by L. I. Mandelstadt (Collection of Complete Works, Vol. II, p. 334, AN USSR ed., 1947).

reasoning and derivations for each of the polarizations are the same, we shall adduce them only for one of the polarizations, for instance for the first parts of formulae (21.2) and (21.3). To simplify writing, we shall temporarily drop the index 0 for the group velocity w_0 and the angle ξ_0.

It is not difficult to see that, for axially-symmetric anisotropy and for the orientation of the surface to be studied (the normal N is directe along the z-axis -- the axis of symmetry), integration with respect to all directions of the wave vector $\vec{p}$ within the limits of the positive unit hemisphere is equivalent to integration with respect to all directions of the group velocity similarly within the limits of a hemisphere -- positive or negative depending on the dispersion law. However, we are also restricted by the condition of realness for p_3 (travelling waves) which, generally speaking, is satisfied not on the entire hemisphere, but only on some parts of it (axially-symmetric cones).

Let us denote by θ and φ the polar angle and the azimuth of the wave vector $\vec{p}$[1]

$$p_1 = P \sin\theta \cos\varphi, \quad p_2 = P \sin\theta \sin\varphi, \quad p_3 = P \cos\theta, \qquad (21.4)$$

and by ξ and ψ the polar angle and the azimuth of the group velocity $\vec{w}$

$$w_1 = w \sin\xi \cos\psi, \quad w_2 = w \sin\xi \sin\psi, \quad w_3 = w \cos\xi. \qquad (21.5)$$

Evidently, for real p_1 and p_2 (in the previous sections we did indeed use real p_1 and p_2) the condition for realness of p_3 reduces to the fact that the angles θ and φ and the modulus of the wave vector P must be real. The domain O introduced by us earlier and determined as the domain of the changing variables p_1 and p_2 in which p_3 is real and positive, therefore is transformed into some domain Ω_1 for real

1. The modulus of the wave vector $\vec{p}$ would also have to be written with the index 0 which we are temporarily omitting.

angles $\theta \leq \pi/2$ and φ (from 0 to 2π), in which P is real. But taking into account the above indicated possibility of cases, when the signs of p_3 and w_3 are different, we must reject the requirement $p_3 \geqslant 0$ and replace it with the condition $w_3 \geqslant 0$. Thus, by Ω_1 we must understand the domain of angles θ and φ corresponding to real P and $w_3 \geqslant 0$. After this it is clear that in the later going over from the integration variables θ and φ to the variables ξ and η the region Ω_1 is transformed precisely into the region Ω_0 which we had introduced in (21.2).

Transforming (21.3) to the form (21.2), we must take into account the scattering equation (19.5), which we shall write in the general form

$$f(\omega, \vec{p}) = 0. \tag{21.6}$$

If we substitute (21.4) here, then this equation can be represented in the form

$$\omega = \omega(P, \theta, \varphi) \qquad P = P(\omega, \theta, \varphi) \tag{21.7}$$

For fixed frequency ω, in view of (21.4) and (21.7), we have

$$dp_1 dp_2 = \frac{\partial(p_1, p_2)}{\partial(\theta, \varphi)} d\theta d\varphi = \begin{vmatrix} \frac{\partial P \sin\theta}{\partial\theta}\cos\varphi & \frac{\partial P \sin\theta}{\partial\theta}\sin\varphi \\ \frac{\partial P \cos\varphi}{\partial\varphi}\sin\theta & \frac{\partial P \sin\varphi}{\partial\varphi}\sin\theta \end{vmatrix} d\theta d\varphi =$$

$$= \frac{1}{2}\left| \frac{\partial (P\sin\theta)^2}{\partial\theta} \right| d\theta d\varphi = \left| P\left(P\cos\theta + \frac{\partial P}{\partial\theta}\sin\theta\right)\right| d\vec{\Omega} \tag{21.8}$$

where $d\vec{\Omega}$ is the element solid angle in the direction of the wave vector $\vec{p}$

$$d\vec{\Omega} = \sin\theta \, d\theta \, d\varphi \tag{21.9}$$

and where we have introduced the absolute value sign, since $dp_1 dp_2 > 0$. Therefore,

$$\int_0 dp_1\, dp_2 = \int_{\Omega_1} \left| P\left(P\cos\theta + \frac{\partial P}{\partial\theta}\sin\theta\right)\right| d\Omega. \qquad (21.10)$$

Now we must go from the solid angle $d\Omega$ to the group (beam) solid angle $d\Omega_0$. For this, $d\Omega_0$ (and at the same time also the group velocity w which will be required later) must be expressed in terms of ω, θ and φ or of P and of its derivatives with respect to ω, θ and φ, since $P(\omega, \theta, \varphi)$ is known from (21.7).

In the determination of the group velocity, we have

$$w_i = \frac{\partial\omega}{\partial p_i} \quad (i = 1,2,3)$$

or, in vector form

$$\vec{w} = \nabla_p \omega .$$

Therefore, in polar coordinates P, θ and φ

$$w_P = \frac{\partial\omega}{\partial P}, \quad w_\theta = \frac{1}{P}\frac{\partial\omega}{\partial\theta}, \quad w_\varphi = \frac{1}{P\sin\theta}\frac{\partial\omega}{\partial\varphi}. \qquad (21.11)$$

Differentiating the scattering equation $P = P[\omega(P, \theta, \varphi)\, \theta, \varphi]$ with respect to P, θ and φ, we obtain

$$1 = \frac{\partial P}{\partial\omega}\frac{\partial\omega}{\partial P}, \quad 0 = \frac{\partial P}{\partial\omega}\frac{\partial\omega}{\partial\theta} + \frac{\partial P}{\partial\theta}, \quad 0 = \frac{\partial P}{\partial\omega}\frac{\partial\omega}{\partial\varphi} + \frac{\partial P}{\partial\varphi}.$$

From here and from (21.11), it follows that

$$w_P = \frac{1}{\frac{\partial P}{\partial\omega}}, \quad w_\theta = -\frac{\frac{1}{P}\frac{\partial P}{\partial\theta}}{\frac{\partial P}{\partial\omega}}, \quad w_\varphi = -\frac{\frac{1}{P\sin\theta}\frac{\partial P}{\partial\varphi}}{\frac{\partial P}{\partial\omega}}, \qquad (21.12)$$

$$w = \sqrt{w_P^2 + w_\theta^2 + w_\varphi^2} = \left| \frac{1}{P\frac{\partial P}{\partial\omega}}\sqrt{P^2 + \left(\frac{\partial P}{\partial\theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial P}{\partial\varphi}\right)^2}\right|. \qquad (21.13)$$

Thus we obtain the expression of the required form for w. It remains for us to express elementary solid angle $d\Omega_0$ in terms of $d\Omega$, P and derivatives of P.

According to (21.5) and (21.9)

$$d\Omega_0 = \sin\xi\, d\xi\, d\eta = \sin\xi \frac{\partial(\xi,\eta)}{\partial(\theta,\varphi)} d\theta\, d\varphi = \frac{\sin\xi}{\sin\theta} \frac{\partial(\xi,\eta)}{\partial(\theta,\varphi)} d\Omega . \tag{21.14}$$

From (21.12) and (21.5), we have

$$\sin\xi\sin\theta\cos(\eta-\varphi) + \cos\xi\cos\theta = \frac{w_P}{w} = \frac{P}{a}$$

$$\sin\xi\cos\theta\cos(\eta-\varphi) - \cos\xi\sin\theta = \frac{w_\theta}{w} = -\frac{1}{a}\frac{\partial P}{\partial\theta} \tag{21.15}$$

$$\sin\xi\sin(\eta-\varphi) = \frac{w_\varphi}{w} = -\frac{1}{a\sin\theta}\frac{\partial P}{\partial\varphi}$$

$$a = \sqrt{P^2 + \left(\frac{\partial P}{\partial\theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial P}{\partial\varphi}\right)^2} .$$

For fixed frequency ω, equations (21.15), of which only two are independent, determine ξ and η as a function of θ and φ and, therefore, permit the determination of the Jacobian contained in (21.14). Carrying out of the evaluation in the general form is necessary, however, only for a biaxial crystal. For media with axial-symmetric anisotropy, if the polar axis is made to coincide with the axis of symmetry, P does not depend on the azimuth φ. Equation (21.13) then takes the form

$$w = \left| \frac{1}{P\,\frac{\partial P}{\partial\omega}} \sqrt{P^2 + \left(\frac{\partial P}{\partial\theta}\right)^2} \right| \tag{21.16}$$

and equations (21.15) give

$$\cos \xi = \frac{P \cos \theta + \frac{\partial P}{\partial \theta} \sin \theta}{\sqrt{P^2 + \left(\frac{\partial P}{\partial \theta}\right)^2}}, \qquad \sin \xi = \frac{\left| P \sin \theta - \frac{\partial P}{\partial \theta} \cos \theta \right|}{\sqrt{P^2 + \left(\frac{\partial P}{\partial \theta}\right)^2}}$$

$$\eta = \begin{cases} \varphi \\ \pi + \varphi \end{cases} \quad \text{for} \quad P \sin \theta - \frac{\partial P}{\partial \theta} \cos \theta \begin{cases} > 0. \\ < 0. \end{cases} \tag{21.17}$$

From (21.17) it follows that

$$\sin \xi \, d\xi = -\frac{\partial}{\partial \theta} \left\{ \frac{P \cos \theta + \frac{\partial P}{\partial \theta} \sin \theta}{\sqrt{P^2 + \left(\frac{\partial P}{\partial \theta}\right)^2}} \right\} d\theta, \qquad d\eta = d\varphi .$$

Thus,

$$d\Omega_0 = \sin \xi \, d\xi \, d\eta = -\frac{\partial}{\partial \theta} \left\{ \frac{P \cos \theta + \frac{\partial P}{\partial \theta} \sin \theta}{\sqrt{P^2 + \left(\frac{\partial P}{\partial \theta}\right)^2}} \right\} \frac{d\Omega}{\sin \theta}. \tag{21.18}$$

Substituting in (21.10) the expression $P \cos \theta + \frac{\partial P}{\partial \theta} \sin \theta$ from (21.17) and the expression for $d\Omega$ in terms of $d\Omega_0$ from (21.18), we obtain ($\xi = \xi_0$, $P = P_0$)

$$\int_0 dp_1 \, dp_2 = \int_{\Omega_0} \left| \frac{P_0 \sqrt{P_0^2 + P_0'^2}}{\frac{\partial}{\partial \theta} \left\{ \frac{P_0 \cos \theta + P_0' \sin \theta}{\sqrt{P_0^2 + P_0'^2}} \right\}} \right| \cos \;_0 \, d \;_0$$

where, as had already been mentioned, the region Ω_0 corresponds to real P_0 and $w_3 \geqslant 0$, and the primes indicate the derivative with respect to θ. If this is introduced into the first term of formula (21.3) and then set identically equal to the first term of (21.2), then we shall convince ourselves that for real P_0

$$I_0 = \frac{I_{0\omega}}{2k^2}\left|\frac{P_0\sqrt{P_0^2 + P_0'^2}\,\sin\theta}{\frac{\partial}{\partial\theta}\left\{\frac{P_0\cos\theta + P_0'\sin\theta}{\sqrt{P_0^2 + P_0'^2}}\right\}}\right|. \qquad (21.19)$$

The intensity I_e is analogously expressed by the second root $P_e(\omega, \theta, \varphi)$ of the scattering equation (21.6). Consequently, the total equilibrium radiation intensity in a homogeneous medium with axial-symmetrical anisotropy is

$$I_\omega = \frac{I_{0\omega}\sin\theta}{2k^2}\left(\left|\frac{P_0\sqrt{P_0^2 + P_0'^2}}{\frac{\partial}{\partial\theta}\left\{\frac{P_0\cos\theta + P_0'\sin\theta}{\sqrt{P_0^2 + P_0'^2}}\right\}}\right| + \right.$$

$$\left. + \left|\frac{P_e\sqrt{P_e^2 + P_e'^2}}{\frac{\partial}{\partial\theta}\left\{\frac{P_e\cos\theta + P_e'\sin\theta}{\sqrt{P_e^2 + P_e'^2}}\right\}}\right|\right). \qquad (21.20)$$

If we introduce the indices of refraction of "ordinary" and "extraordinary" waves

$$P_0 = kn_0, \quad P_e = kn_e, \qquad (21.21)$$

then (21.20) takes the form

$$I_\omega = \frac{I_{0\omega}\sin\theta}{2}\left(\left|\frac{n_0\sqrt{n_0^2 + n_0'^2}}{\frac{\partial}{\partial\theta}\left\{\frac{n_0\cos\theta + n_0'\sin\theta}{\sqrt{n_0^2 + n_0'^2}}\right\}}\right| + \right. \qquad (21.22)$$

$$+ \left| \frac{n_e \sqrt{n_e^2 + n_e'^2}}{\frac{\partial}{\partial \theta} \frac{n_e \cos \theta + n_e' \sin \theta}{\sqrt{n_e^2 + n_e'^2}}} \right| \Bigg) . \qquad (21.22)$$

For the case of an isotropic medium for which the indices of refraction $n = n_0 = n_e$ do not depend on θ, we obtain from (21.22) the first formula of (18.1)

$$I_\omega = I_{0\omega} n^2 .$$

Having the expressions for the intensities I_0 and I_e at our disposal, we can now transform formula (21.1) for the energy density. Substituting into it I_0 from (21.19), $w \equiv w_0$ from (21.16), $d\Omega_0$ from (21.18), and similarly for the analogous expressions for I_e, w_e and $d\Omega_e$, and taking into account that $I_{0\omega} = \frac{c}{4\pi} u_{0\omega}$, we obtain

$$u_\omega = \frac{u_{0\omega} c}{8\pi k^2} \left\{ \int_{\Omega_1} P_0^2 \left| \frac{\partial P_0}{\partial \omega} \right| d\Omega + \int_{\Omega_2} P_e^2 \left| \frac{\partial P_e}{\partial \omega} \right| d\Omega \right\}$$

or, in terms of the indices of refraction n_0 and n_e

$$u_\omega = \frac{u_{0\omega}}{8\pi} \left\{ \int_{\Omega_1} n_0^2 \left| \frac{\partial n_0 \omega}{\partial \omega} \right| d\Omega + \int_{\Omega_2} n_e^2 \left| \frac{\partial n_e \omega}{\partial \omega} \right| d\Omega \right\} . \qquad (21.23)$$

Here Ω_1 and Ω_2 are the regions of the directions of the wave vector, corresponding to those regions Ω_0 and Ω_e of the directions of the group velocity, in which n_0 and n_e are, respectively, real; and here all regions are taken within the limits of the entire unit sphere.

For an isotropic medium ($\Omega_1 = \Omega_2 = 4\pi$), we obtain

$$u_\omega = u_{0\omega} n^2 \left| \frac{\partial n \omega}{\partial \omega} \right| . \qquad (21.24)$$

In the absence of scattering (n independent of ω) this goes over into the second formula of (18.1)

$$u_\omega = t_\omega n^3 .$$

Section 22. Classical Derivation of the Formula for Energy Density

The asymptotic expression (21.23) was obtained as a result of the solution of the problem of equilibrium radiation in a plane wave guide with a "noisy" wall. Let us now show that the classical method indicated in Section 18, based on the estimation of the number dZ of natural field oscillations in a sufficiently large volume V, leads to the same expression.

Let some region of space be filled with an homogeneous transparent anisotropic medium. It is easiest of all to imagine the boundaries of this region to be ideally reflecting walls but, generally speaking, this is not necessary. For very general assumptions concerning the properties of the medium, the shape of the region and the form of the boundary conditions, it can be shown that the asymptotic distribution of the natural values in the space of wave vectors $\vec{p}$ (distributed for large p) is uniform.[1] In other words, in the coordinates p_1, p_2, p_3, the natural values are represented by the nodes of the space grid whose elementary cell volume, in the region of sufficiently large p_1, p_2, p_3, is constant and equal to π^3/V, where V is the volume of the region under consideration. Thus, the number of natural oscillations, included in a volume of the $\vec{p}$-space $\int dp_1\, dp_2\, dp_3$ and corresponding to any one of the two independent polarizations possible in the given medium, is asymptotically equal to

$$Z = \frac{V}{\pi^3} \int dp_1\, dp_2\, dp_3 .$$

To find the equilibrium electromagnetic energy density in the spectral interval $(\omega, \omega + d\omega)$, one must evaluate the number dZ of natural oscillations (of both polarizations) whose frequencies lie in this

1. R. Courant and D. Gilbert, Methods of Mathematical Physics, Vol. I, Chapt. VI, Section 4 (M.-L., 1951).

interval. To each of these oscillations corresponds the energy $\mathcal{E}(\omega, T)$ -- the mean energy of the oscillator of frequency ω at temperature T, given by formula (5.1). Thus,

$$u_\omega d\omega = \mathcal{E}(\omega, T)\frac{dZ}{V} .$$

Since the equilibrium energy density in vacuum is related to $\mathcal{E}(\omega, T)$ by the relation [see (5.1) and (5.3)]

$$u_{0\omega} = \frac{k^2}{\pi^2 c}\mathcal{E}(\omega, T),$$

the preceding formula can be written in the form

$$u_\omega d\omega = u_{0\omega}\frac{\pi^2 c}{k^2}\frac{dZ}{V} . \qquad (22.1)$$

Let us introduce into the $\vec{p}$-space the polar coordinates

$$p_1 = P\sin\theta\cos\varphi, \quad p_2 = P\sin\theta\sin\varphi, \quad p_3 = P\cos\theta . \qquad (22.2)$$

If, instead of p_1, p_2, p_3, we take ω, θ and φ as the independent variables, then we obtain for the elementary volume of the $\vec{p}$-space

$$dp_1\, dp_2\, dp_3 = \frac{\partial(p_1, p_2, p_3)}{\partial(\omega, \theta, \varphi)}\, d\theta\, d\varphi\, d\omega,$$

where the Jacobian entering here will be, according to (22.2) and (21.7),

$$\frac{\partial(p_1, p_2, p_3)}{\partial(\omega, \theta, \varphi)} = \begin{vmatrix} \frac{\partial P}{\partial\omega}\sin\theta\cos\varphi & \frac{\partial P}{\partial\omega}\sin\theta\sin\varphi & \frac{\partial P}{\partial\omega}\cos\theta \\ \frac{\partial P\sin\theta}{\partial\theta}\cos\varphi & \frac{\partial P\sin\theta}{\partial\theta}\sin\varphi & \frac{\partial P\cos\theta}{\partial\theta} \\ \frac{\partial P\cos\varphi}{\partial\varphi}\sin\theta & \frac{\partial P\sin\varphi}{\partial\varphi}\sin\theta & \frac{\partial P}{\partial\varphi}\cos\theta \end{vmatrix} =$$

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THEORY OF ELECTRIC FLUCTUATIONS AND THERMAL RADIATION

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Let us note that the conditions for the realness of w_0 and w_e coincide with the conditions for the realness of the corresponding P_0 and P_e.

Let us initially admit $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$. In this case P_0 and P_e are real for all θ, $\gamma_{0,e} = \varphi$ and, since a change in θ from 0 to $\pi/2$ corresponds to a change in ξ_0 and ξ_e within the same limits, the regions Ω_0 and Ω_e coincide with the regions Ω_1 and Ω_2.

Substituting (23.3) into the formula (21.20) for the intensity, we get

$$I_\omega = \frac{I_{0\omega}}{2}\left\{\varepsilon_1 + \frac{1}{\varepsilon_1}\left(\frac{\varepsilon_1^2 \sin^2\theta + \varepsilon_3^2 \cos^2\theta}{\varepsilon_1 \sin^2\theta + \varepsilon_3 \cos^2\theta}\right)^2\right\} . \qquad (23.6)$$

The first term gives the equilibrium intensity of ordinary waves, the second that of the extraordinary ones. For $\varepsilon_1 = \varepsilon_3 = \varepsilon$ (isotropic medium) we obtain the usual expression $I_\omega = I_{0\omega}\varepsilon = I_{0\omega}n^2$. Carrying (23.3) into (21.22) we find for the equilibrium energy density

$$u_\omega = \frac{u_{0\omega}}{8\pi k^3} \oint (P_0^3 + P_e^3)\, d\Omega = \qquad (23.7)$$

$$= \frac{u_{0\omega}}{4} \int_0^\pi \left\{\varepsilon_1^{3/2} + \left(\frac{\varepsilon_1\varepsilon_3}{\varepsilon_1 \sin^2\theta + \varepsilon_3 \cos^2\theta}\right)^{3/2}\right\} \sin\theta\, d\theta =$$

$$= \frac{u_{0\omega}}{2}\left(\varepsilon_1^{3/2} + \varepsilon_3\varepsilon_1^{1/2}\right) .$$

When $\varepsilon_1 = \varepsilon_3 = \varepsilon$, this goes over into the usual formula for the isotropic, non-scattering medium $u_\omega = u_{0\omega}\varepsilon^{3/2} = u_{0\omega}n^3$.

Let us now turn to the cases (not realizable in the common crystals, but presenting interest for what is to follow), when the components ε_1 and ε_3 have different signs or when both are negative.

Let $\varepsilon_1 < 0$ and $\varepsilon_3 > 0$. In this case P_0 is imaginary for all angles,

so that the intensity and density of energy of ordinary waves are equal to zero. The second root p_e is real for $\theta_{min} \leqslant \theta \leqslant \pi - \theta_{min}$, where $\theta_{min} = \mathrm{arctg}\sqrt{\frac{\varepsilon_3}{|\varepsilon_1|}}$. In this region of values of θ, according to (23.3) and (23.5),

$$p_e = kn_e = k\sqrt{\frac{|\varepsilon_1|\varepsilon_3}{|\varepsilon_1|\sin^2\theta - \varepsilon_3\cos^2\theta}},$$

$$w_e = c\sqrt{\frac{\varepsilon_1^2\sin^2\theta + \varepsilon_3^2\cos^2\theta}{|\varepsilon_1|\varepsilon_3(|\varepsilon_1|\sin^2\theta - \varepsilon_3\cos^2\theta)}} \qquad \varphi_e = \varphi$$

and ξ_e varies from ξ_{min} to $\pi - \xi_{min}$, where $\xi_{min} = \mathrm{arctg}\sqrt{\frac{\varepsilon_1}{|\varepsilon_3|}}$. Figure 10 shows how p_e and w_e vary in magnitude and in direction when θ varies within the indicated limits. The total intensity, which in the case under study reduces to the intensity of the extraordinary waves, is expressed as follows:

$$I_\omega = \begin{cases} \dfrac{I_0}{2|\varepsilon_1|}\left(\dfrac{\varepsilon_1^2\sin^2\theta + \varepsilon_3^2\cos^2\theta}{|\varepsilon_1|\sin^2\theta - \varepsilon_3\cos^2\theta}\right)^2 & \text{for} \quad \theta_{min} < \theta < \pi - \theta_{min} \\ 0 \quad \text{for} \quad 0 \leqslant \theta < \theta_{min} & \text{and} \quad \pi - \theta_{min} < \theta \leqslant \pi. \end{cases}$$

As θ approaches the limiting angles θ_{min} and $\pi - \theta_{min}$, the intensity increases so rapidly that the expressions for the flow and the density of energy diverge.

If $\varepsilon_1 > 0$ and $\varepsilon_3 < 0$, then the intensity of ordinary waves is expressed as before, as in (23.6). p_e is now real when $0 \leqslant \theta \leqslant \theta_{max}$ and $\pi - \theta_{max} \leqslant \theta \leqslant \pi$, where $\theta_{max} = \mathrm{arctg}\sqrt{|\varepsilon_3|/\varepsilon_1}$. For the angles θ, lying within these limits, we have

$$p_e = kn_e = k\sqrt{\frac{\varepsilon_1|\varepsilon_3|}{|\varepsilon_3|\cos^2\theta - \varepsilon_1\sin^2\theta}},$$

$$w_e = c\sqrt{\frac{\varepsilon_1^2\sin^2\theta + \varepsilon_3^2\cos^2\theta}{\varepsilon_1|\varepsilon_3|(|\varepsilon_3|\cos^2\theta - \varepsilon_1\sin^2\theta)}}, \quad \eta_e = \pi + \varphi,$$

where a change in ξ from π to $\pi - \xi_{max}$ corresponds to a change in θ from 0 to θ_{max}, with $\xi_{max} = \operatorname{arctg}\sqrt{\frac{|\varepsilon_3|}{\varepsilon_1}}$. Thus, we have here precisely that case for which the group velocity is negative, i.e., to the direction p_e with $p_{e3} > 0$ corresponds the direction w_e in the direction of the negative hemisphere (Fig. 11). We obtain for the intensity

$$I_\omega = \begin{cases} \frac{I_{0\omega}}{2}\left\{\varepsilon_1 + \frac{1}{\varepsilon_1}\left(\frac{\varepsilon_1^2\cos^2\theta + \varepsilon_3^2\cos^2\theta}{|\varepsilon_3|\cos^2\theta - \varepsilon_1\sin^2\theta}\right)^2\right\} & \text{for } 0 \leqslant \theta \leqslant \theta_{max} \\ & \text{and } \pi - \theta_{max} \leqslant \theta \leqslant \pi. \\ \frac{I_{0\omega}}{2}\varepsilon_1 \quad \text{for} \quad \theta_{max} < \theta < \pi - \theta_{max}. \end{cases}$$

The energy flow and the energy density of extraordinary waves are similarly for this case infinite.

Finally, for $\varepsilon_1 < 0$ and $\varepsilon_3 < 0$ both P_0 and P_e (i.e., the indices of refraction n_0 and n_e) are always imaginary. Correspondingly, $I_\omega = 0$ and $u_\omega = 0$.

Despite the fact that the coefficients of the scattering equation (23.1) for a magnetoactive medium are only a little more complex than the coefficients of equation (23.2) for an uni-axial crystal, all relations and formulae obtainable with the help of (23.1) are much more complicated and unwieldy.

The roots of equation (23.1), which become the roots P_0 and P_e, respectively, of equation (23.2) for $\beta = 0$ and $\alpha_1(\alpha_1 - \alpha_3) > 0$,

are expressed as follows

$$P_{0,e}^2 = \frac{[\alpha_1(\alpha_1-\alpha_3)-\beta^2]\sin^2\theta + 2\alpha_1\alpha_3 \pm \sqrt{[\alpha_1(\alpha_1-\alpha_3)-\beta^2]^2\sin^4\theta + 4\alpha_3^2\beta^2\cos^2\theta}}{2[(\alpha_1-\alpha_3)\sin^2\theta + \alpha_3]} . \tag{23.8}$$

Using these roots and the equation (23.1) itself, one can obtain the following formula for the equilibrium intensity (see Appendix VIII) with the help of (21.19):

$$I_0 = \frac{I_{0\omega}P_0^2}{2k^2} \times$$

$$\times \frac{(D^2 + B^2\sin^2\theta\cos^2\theta)^2}{|(D - B\cos^2\theta)[D(D - B\cos^2\theta)(D + B\sin^2\theta) + 8A^2P_0^2BC\sin^2\theta\cos^2\theta]|}$$

$$I_e = \frac{I_{0\omega}P_e^2}{2k^2} \times \tag{23.9}$$

$$\times \frac{(D^2 + C^2\sin^2\theta\cos^2\theta)^2}{|D + C\cos^2\theta)[D(D + C\cos^2\theta)(D - C\sin^2\theta) - 8A^2P_e^2BC\sin^2\theta\cos^2\theta]|}$$

where

$$A = (\alpha_1 - \alpha_3)\sin^2\theta + \alpha_3,$$

$$\left.\begin{matrix} B \\ C \end{matrix}\right\} = [\alpha_1(\alpha_1-\alpha_3) - \beta^2](\alpha_1 - \alpha_3)\sin^2\theta - 2\alpha_3\beta^2 \mp (\alpha_1 - \alpha_3)E, \tag{23.10}$$

$$D = 2AE, \quad E = \sqrt{[\alpha_1(\alpha_1-\alpha_3) - \beta^2]^2\sin^4\theta + 4\alpha_3^2\beta^2\cos^2\theta} .$$

It is not difficult to convince oneself, that for $\beta = 0$ the expressions

(23.9) become respectively the first and second terms of formula (23.6) for the intensity of equilibrium radiation in an uni-axial crystal. Still, in examining radiation in a magnetoactive medium it is expedient to uncover the coefficients α_1, α_3 and β, with the aim of expressing all magnitudes in terms of parameters which characterize directly such a medium, namely the natural frequency of the plasma in the absence of the magnetic field (ω_0) and the gyromagnetic frequency (ω_H).

According to (20.15), we have

$$\alpha_1 = k^2\left(1 - \frac{\omega_0^2}{\omega^2 - \omega_H^2}\right), \quad \alpha_3 = k^2\left(1 - \frac{\omega_0^2}{\omega^2}\right), \quad \beta = k^2\,\frac{\omega_0^2\omega_H}{(\omega^2 - \omega_H^2)\,\omega}.$$

Let us use the gyromagnetic frequency ω_H as a unit and let us introduce the notation[1]

$$\zeta = \frac{\omega}{\omega_H}, \quad \alpha = \frac{\omega_0}{\omega_H}, \quad \varkappa = \frac{\omega_H}{c}. \tag{23.11}$$

Then

$$\alpha_1 = \varkappa^2\,\frac{\zeta^2(\zeta^2 - \alpha^2 - 1)}{\zeta^2 - 1}, \quad \alpha_3 = \varkappa^2(\zeta^2 - \alpha^2), \quad \beta = \varkappa^2\,\frac{\alpha^2\zeta}{\zeta^2 - 1}. \tag{23.12}$$

We shall further assume that the natural frequency ω_0 is less than the gyromagnetic frequency ω_H, i.e., $\alpha < 1$. With this assumption it is not possible to examine the passage to the absence of the magnetic field, since we have $\omega_H \to 0$ and $\alpha \to \infty$ when $\mathcal{H} \to 0$; nevertheless the assumption corresponds to concrete conditions in the ionosphere. In the E layer of the ionosphere at an electron concentration of $N = 10^5$

1. In the ionosphere $\mathcal{H} \sim 0.5$ oersteds. To the frequency ω_H corresponds for this the wavelength of about 214 m. Let us note, that the parameters ζ and α introduced by us are related to the parameters u and v which are used by V. L. Ginzburg in his book "Theory of Radio-wave Propagation in the Ionosphere" in the following way: $\alpha^2 = \frac{v}{u}$, $\zeta^2 = \frac{1}{u}$.

the wavelength corresponding to $\omega_{\bar{}}$ constitutes about 330 m, so that $\alpha \approx 2/3$.

If we substitute (23.12) into expression (23.8) for the roots of the scattering equation, we obtain

$$F^2_{0,e} = \frac{\varkappa^2\zeta^2\{\alpha^2 \sin^2\theta - 2(\zeta^2 - \alpha^2)(\zeta^2 - \alpha^2 - 1)\}}{2\{\alpha^2 \sin^2\theta - (\zeta^2 - \alpha^2)(\zeta^2 - 1)\}} \mp$$

$$\mp \frac{\alpha^2 \operatorname{sgn}(\zeta^2 - 1)\sqrt{\sin^4\theta + \frac{4(\zeta^2 - \alpha^2)^2}{\zeta^2}\cos^2\theta}}{2\{\alpha^2 \sin^2\theta - (\zeta^2 - \alpha^2)(\zeta^2 - 1)\}} .$$

Multiplying numerator and denominator by the multiplier which complements the numerator up to the difference in squares, we rid ourselves of the radical in the numerator and at the same time we bring the expressions for F^2_0 and F^2_e to the form which contains the angle θ only in the denominator

$$F^2_{0,e} = -\frac{2\varkappa^2(\zeta^2 - \alpha^2)(\zeta^4 - (2\alpha^2 + 1)\zeta^2 + \alpha^4)}{\alpha^2 \sin^2\theta - 2(\zeta^2 - \alpha^2)(\zeta^2 - \alpha^2 - 1) \pm \alpha^2 \operatorname{sgn}(\zeta^2 - 1)\sqrt{}}$$

where

$$\sqrt{} = \sqrt{\sin^4\theta + \frac{4(\zeta^2 - \alpha^2)^2}{\zeta^2}\cos^2\theta} . \tag{23.13}$$

Let us remember that division of $F^2_{0,e}$ by $k^2 = \varkappa^2\zeta^2$ gives the squares of the indices of refraction $n^2_{0,e}$.

The numerator of (23.13) becomes zero for the following values of ζ^2 which we present in the increasing order:

$$\zeta^2_1 = \frac{2\alpha^2 + 1 - \sqrt{4\alpha^2 + 1}}{2}, \quad \alpha^2, \quad \zeta^2_2 = \frac{2\alpha^2 + 1 + \sqrt{4\alpha^2 + 1}}{2}. \tag{23.14}$$

The behavior of the denominators of F_0^2 and F_e^2 is more complex. Namely, the denominator of F_0^2 is negative in the intervals $\zeta^2(0,\alpha^2)$ and (ζ_2^2, ∞); in the interval (α^2, ζ_2^2) it is positive. The denominator of F_e^2 is negative in the interval $\zeta^2(1+\alpha^2, \infty)$, positive in $(\alpha^2,1)$, and in the intervals $(0,\alpha^2)$ and $(1, 1+\alpha^2)$ this denominator can go through zero for the angle θ_0 given by

$$\sin^2\theta_0 = \frac{(\zeta^2-\alpha^2)(\zeta^2-1)}{\alpha^2}. \qquad (23.15)$$

Taking into account that $\zeta_1^2 < \alpha^2 < 1 < 1+\alpha^2 < \zeta_2^2$, we obtain in summary the following distribution of signs for F_0^2 and F_e^2 (and therefore also for the squares of the indices of refraction n_0^2 and n_e^2):

Interval of values for ζ^2	Sign of F_0^2	Sign of F_e^2
$\zeta^2 < \zeta_1^2$	-	+ for $\theta \leqslant \theta_0$
$\zeta_1^2 < \zeta^2 < \alpha^2$	+	+ for $\theta \leqslant \theta_0$
$\alpha^2 < \zeta^2 < 1$	+	+
$1 < \zeta^2 < 1+\alpha^2$	+	+ for $\theta \leqslant \theta_0$
$1+\alpha^2 < \zeta^2 < \zeta_2^2$	+	-
$\zeta_2^2 < \zeta^2$	+	+

The intensities of equilibrium radiation I_0 and I_e are different from zero only in the regions of real n_0 and n_e (non-extinguishing waves). Therefore, we have $I_0 = 0$ in the interval $\zeta^2 < \zeta_1^2$, and $I_e = 0$ in the

interval $1 + \alpha^2 < \zeta^2 < \zeta_2^2$ in which n_e^2 is negative. At sufficiently high frequencies ($\zeta \gg \zeta_2$), according to (23.13), $F_0^2 \approx F_e^2 \approx \chi^2\zeta^2 = k^2$, i.e., the medium behaves like a vacuum and, correspondingly $I_0 = I_e = I_{0\omega}/2$.

Substitution of (23.12) into (23.10) gives the following expression for the quantities A, B, etc., appearing in the formula for the intensities (23.9):

$$A = -\frac{\chi^2\alpha^2}{\zeta^2 - 1}\left\{\sin^2\theta - \frac{(\zeta^2 - \alpha^2)(\zeta^2 - 1)}{\alpha^2}\right\}$$

$$\left.\begin{matrix} B \\ C \end{matrix}\right\} = \frac{\chi^6\alpha^4\zeta^2}{(\zeta^2 - 1)^2}\left\{\sin^2\theta - 2(\zeta^2 - \alpha^2) \pm R_1\right\} \qquad (23.16)$$

$$D = -\frac{2\chi^6\alpha^4\zeta^2}{(\zeta^2 - 1)^2}\left\{\sin^2\theta - \frac{(\zeta^2 - \alpha^2)(\zeta^2 - 1)}{\alpha^2}\right\} R_1$$

$$R_1 = \operatorname{sgn}(\zeta^2 - 1)\sqrt{\sin^4\theta + \frac{4(\zeta^2 - \alpha^2)^2}{\zeta^2}\cos^2\theta}.$$

With the aid of these expressions and formula (23.9), we shall construct graphs for the angular equilibrium intensity distributions $\frac{I_0}{I_{0\omega}}$ and $\frac{I_e}{I_{0\omega}}$ (Fig. 12-22). On the right is shown the trend of $\frac{I_0}{I_{0\omega}}$, on the left the trend of $\frac{I_e}{I_{0\omega}}$. The angle θ is reckoned from the direction of the magnetic field $\mathcal{H}$. With respect to the plane $\theta = 90°$ the trend of I_0 and I_e is symmetrical. The graphs are constructed for $\alpha = \frac{2}{3}(\omega_0 = \frac{2}{3}\omega_H)$, corresponding to $\zeta_1^2 = \frac{1}{9}$, $\alpha^2 = \frac{4}{9}$, $1 + \alpha^2 = \frac{13}{9}$, $\zeta_2^2 = \frac{16}{9}$. These numbers (as well as $\zeta^2 = 1$) are the limits of intervals shown in the preceding table. The graphs are constructed for these boundary values of ζ^2 and for the interval values $\zeta^2 = \frac{1}{18}, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}$ and 2. On each graph are given along with the value of ζ^2 the

corresponding relation (in brackets) between the wave frequency ω and the frequencies ω_0 and ω_H characterizing the medium. In the lower parts of the drawings is shown the trend of (real) indices of refraction n_0 and n_e.

For $\zeta^2 \leqslant \frac{1}{9}$, the emission is completely polarized, since $I_0 = 0$; I_e is different from zero only for the limiting angle θ_0, which at $\zeta^2 = \frac{1}{18}$ is 65.2° and decreases with increasing ζ^2 to zero for $\zeta^2 = \frac{4}{9} - 0$. Beginning with $\zeta^2 = \frac{1}{9}$ the intensity I_0 appears and is different from zero for all angles θ. When increasing values of ζ^2 approach the value $\zeta^2 = \frac{4}{9}$ (i.e., $\omega \to \omega_0$), the intensity I_0 tends toward the same distribution $I_0 = I_{0\omega}/2$ as in vacuum and the intensity I_e decreases to zero everywhere except in the narrow cone drawing together toward the direction of the magnetic field. In the transition through $\zeta^2 = \frac{4}{9}$ the picture changes stepwise, since I_e too assumes a uniform distribution $I_e = I_{0\omega}/2$; this case ($\zeta^2 = \frac{4}{9} + 0$) is presented in Fig. 15.

For further increases in ζ^2 both intensities are different from zero for all angles θ right up to $\zeta^2 = 1$. In crossing this value the minimum limiting angle θ_0 appears in I_e; this angle grows from zero for $\zeta^2 = 1$ to 90° for $\zeta^2 = 1 + \zeta^2 = \frac{13}{9}$ (i.e., for $\omega = \sqrt{\omega_0^2 + \omega_H^2}$). Also, up to $\zeta^2 = \zeta_2^2 = \frac{16}{9}$ the emission is again completely polarized, since only $I_0 \neq 0$. Starting with $\zeta^2 = \frac{16}{9}$, I_e again appears, and now both intensities are finite for all angles and, as ζ increases (i.e., as the frequency ω increases), they tend to the equilibrium distribution $I_0 = I_e = I_{0\omega}/2$.

The infinite peaks of I_0 and I_e have the following origin. They occur at the angles $\theta = \theta_c$ for which the denominators in formulas (23.9) become zero. In accordance with (21.17) and (21.19), these denominators represent nothing else but $\left|\frac{d\cos\xi}{d\theta}\right| = \sin\theta \left|\frac{d\xi}{d\theta}\right|$. Therefore, zero denominators and, correspondingly, infinite peaks of the intensities,

correspond to either such directions of the wave vector (the angle ϑ) for which the group velocity is directed along the magnetic field ($\xi = 0$), or to the angle ϑ for which $\frac{d\xi}{d\vartheta} = 0$, i.e., a change in the direction of the wave normal does not entrain a turning of the group velocity vector. The beam solid angle element $d\Omega_0$ is related to the normal solid angle element $d\Omega$ by the evident equation:

$$d\Omega_{0,e} = \frac{\sin \xi}{\sin \vartheta} \frac{d\xi}{d\vartheta} \, d\Omega .$$

Thus, for the angle θ_{∞} for which $\xi = 0$ or $\frac{d\xi}{d\theta} = 0$, a thickening of the rays is produced which causes the intensity in the cone formed with the angle θ_{∞} to be infinitely great. On the graphs the intensity peaks caused by $\xi = 0$ or $\frac{d\xi}{d\theta} = 0$ are denoted by the signs 0 or x, respectively.

It can be seen from the graphs that whenever the curve of the refraction index goes to infinity[1], the equilibrium intensity is also infinitely large and is caused by the fact that the derivative $\frac{d\xi}{d\theta}$ becomes zero. But in some intervals of ζ infinite peaks of intensity occur also for such angles θ, for which the trend of the refraction index is plane (I_0 on Fig. 14, 16, 17, 21; I_e on Fig. 12 and 13).

To evaluate the equilibrium energy density one must substitute in (21.23) the expressions for n_0^2 and n_e^2 taken from (23.13). This gives very complicated integrals which cannot be evaluated in closed form. One can however assert that, for finite n_0^2 and n_e^2, i.e., for all cases when there is no critical angle θ_0 (intervals $\alpha^2 < \zeta^2 < 1$ and $1 + \alpha^2 < \zeta^2$), the value of u_ω also is finite. In the presence of the critical angle θ_0, the absolute magnitude of the corresponding index of refraction increases as $|\theta - \theta_0|^{-1/2}$ when $\theta \to \theta_0$, but the behavior of the integrands in (21.23), containing also the derivatives $\frac{\partial n_0 \omega}{\partial \omega}$ and $\frac{\partial n_e \omega}{\partial \omega}$,

1. This is possible because of the absence of absorption. In this, n^2 changes by a jump from $+\infty$ to $-\infty$ so that further on n is imaginary.

requires additional examination. In connection with this, it should be noted that the intervals $\zeta^2 < \alpha^2$ and $1 < \zeta^2 < 1 + \alpha^2$ in which for Σ_0 the limiting angle θ_0 occurs, are exactly those intervals where the signs of the coefficients α_1 and α_3 are distinct [see (23.12)]. In the case of the uni-axial crystal the difference in signs of α_1 and α_3 gave, as we have seen, also a difference for the critical angle in the distribution of the intensity and this caused there an infinitely large value of the energy density. We can expect in the case of a magnetoactive medium under study that the energy density (as also in the case of an isotropic medium) will, in the presence of infinite branches for the refractive index, also be infinitely large. In the absence of absorption, when the resonance increase of the oscillation energy of microparticles is not limited by anything, such a result does not contain in itself any contradictions whatever.

Let us note in conclusion that in practice one does not deal with equilibrium radiation within a magnetoactive medium but with the radiation of such a medium to the exterior (radiation of ionosphere, sun spots). The results studied here are of interest mainly because they can be of advantage in the examination of this last question which in the present work we shall not touch.

CHAPTER V. ELECTRIC FLUCTUATIONS IN A QUASI-STATIONARY REGION

Section 24. Formulation of the Problem

In the region of low frequencies ω, when the dimensions of the bodies under study are much smaller than the wavelength in the surrounding space, the nature of the questions asked of theory changes. As a rule, our interest lies no longer in radiation but in the fluctuations of those integral magnitudes with which the state of electric systems in the quasi-stationary region can be described. In practice, of special importance is the case of systems which represent electric networks of sufficiently thin (quasi-linear) wires, characterized by concentrated impedances. We then speak of fluctuations of the total current strength in any branch of such a network or of fluctuations of the potential on these or those of its impedances.

The theory of quasi-stationary currents in networks with concentrated parameters, operating with the generalized Kirchhoff equations for these currents, is a consequence of the general field equations. But the theory of fluctuations in such networks follows from electrodynamic operations containing the lateral fluctuation field $\vec{K}$. The presence of this field causes the appearance in the corresponding Kirchhoff equations of fluctuating electro-propulsive forces localized in the active resistances of the network. Thus, the fluctuations of all the integral characteristics of the network state -- strength of currents, charges or potentials -- are completely determined by the impedances of the network and the integral fluctuation emf's. Therefore the question of greatest physical interest is the question concerning the relation between the integral random emf's which enter the Kirchhoff equations for alternating currents and the potential which determines their lateral fluctuating field $\vec{K}$ distributed over the volume of the conductors which constitute the network.

If the conductance σ exceeds by many times the frequency ω, then one can neglect the current perturbations inside the conductors. We then have

$$\varepsilon = \varepsilon' - i\varepsilon'' \approx -i\frac{4\pi\sigma}{\omega}, \quad |\varepsilon| \gg 1 \qquad (24.1)$$

and the basic field equations inside the conductors

$$\mathrm{curl}\ \vec{E} = -ik\mu\vec{H}$$

$$\mathrm{curl}\ \vec{H} = ik\varepsilon\vec{E} + ik(\varepsilon - 1)\vec{K}$$

assume the form[1]

$$\mathrm{curl}\ \vec{E} = -ik\mu\vec{H}$$

$$\mathrm{curl}\ \vec{H} = \frac{4\pi}{c}\vec{j} \qquad (24.2)$$

$$\vec{j} = \sigma(\vec{E} + \vec{K}) .$$

Here the value of the constant C in the correlation function of the components of the lateral field $\vec{K}$, as long as we can consciously consider $\hbar\omega \ll \theta$, i.e., starting from the Rayleigh-Jeans law, will be [see (6.17)]

$$C = \frac{4\theta}{\omega}\,\mathrm{Im}\left(\frac{1}{\varepsilon - 1}\right) \approx \frac{4\theta}{\omega}\,\mathrm{Im}\,\frac{1}{\varepsilon} \approx \frac{4\theta}{\omega}\,\frac{\omega}{4\pi\sigma} = \frac{\theta}{\pi\sigma} . \qquad (24.3)$$

It is this value of C which had been previously found in the work of M. A. Leontovich and of the author[2] in the studies of electric fluctuations and of correlation functions of $\vec{K}$ in the quasi-stationary region.

The determination of the spectral intensity of the integral emf $\mathcal{E}$ is divided into two stages. The first stage is purely electrodynamic. It is necessary to establish how the emf $\mathcal{E}$, acting in a part of the network (or in the entire closed network), is expressed in terms of the lateral field $\vec{K}$, arbitrarily distributed over the entire volume of the conductors constituting the given part of the network (or the entire net-

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1. Let us note that as a result of neglecting the perturbations by the currents the lateral field $\vec{K}$ now enters the simple Ohm's law and, therefore, coincides with the lateral field $\vec{K}_0$ which we have previously discussed in Sections 1 and 6.

2. M. A. Leontovich and S. M. Rytov, J.E.T.F. 23, 246, 1952.

work). If such an expression is found, then there remains a second part of the problem -- the establishment of the form of the space correlation function of the components of $\vec{K}$. Having the expression for $\mathcal{E}$ in terms of $\vec{K}$ and knowing the correlation function of $\vec{K}$, we can immediately find the spectral intensities of the fluctuations of the emf $\mathcal{E}$ as well as of the magnitudes determined by it (current strength etc.)

In the cited work, the expression for $\mathcal{E}$ in terms of $\vec{K}$ had been derived for the case of a network of very special form, namely -- for a closed helix of round cylindrical wire. Concerning the correlation function for the longitudinal (axis along the wire) component of $\vec{K}$: in the assumption that the δ-correlation takes place, the coefficient C was obtained from Nyquist's formula for the spectral intensity of the integral emf $\mathcal{E}$

$$\mathcal{E}_{\omega}^{2} = \frac{2\theta}{\pi} R . \tag{24.4}$$

Now it is possible for us to go backwards, since, as a result of the investigation of thermal radiation, the correlation function of the components of $\vec{K}$, including the value (24.3) for the correlation constant C, are already known. Thus also, we need only the expression of $\mathcal{E}$ in terms of $\vec{K}$ in order to find $\mathcal{E}_{\omega}^{2}$. In the next section we shall give the solution of this last problem in the general form, following, with minor changes, the derivation of A. V. Gaponov.

Let us note again that the admittance of the δ-correlation for the components of $\vec{K}$ relies on the smallness of the factual correlation radius a in comparison to the cross-sectional area of the conductors as well as to skin-layer thickness

$$d = \frac{c}{\sqrt{2\pi\sigma\mu\omega}} . \tag{24.5}$$

As far as the relation between the cross-sectional dimensions and d is concerned, it is not fixed in any way, i.e., the results must embrace the case of uniform current distribution over the section (sufficiently low frequencies) as well as the case of skin-effects of any strength when

d is much smaller than the section dimensions (but, as before, d >> a).

Section 25. Expression for Integral emf

For simplicity we shall first study an unramified network so that the equation for the total quasi-stationary current I must be simply the integral Ohm's law. It can be obtained by using the energy determinations of the resistance and of the total inductance of the network. These magnitudes depend on electric and geometric parameters of the conductors, as well as on the current distribution in them, i.e., in final accounting -- on the distribution of the lateral field $\vec{K}$. The magnitude representing the integral emf also proves to be dependent on $\vec{K}$ not only explicitly, but through the current distribution determined by the given $\vec{K}$. It is evident that such a determination of the integral emf is not useful for the problems interesting us, since it does not make it possible to find the spectral intensity of the emf with the help of the correlation function components of $\vec{K}$. Nevertheless we shall adduce the appropriate derivation since the latter helps to narrow down the problem formulation.

Equation (24.2) can be extended to the entire space assuming that outside the volume V of the conductors (in vacuum) $\mu = 1$ and $\sigma = 0$. We thereby completely exclude from the examination currents of perturbation in the external field (outside the conductors curl $\vec{H} = 0$) and, therefore, we exclude the presence of condensers in the network, i.e., we are examining a closed conducting network.

The passing from (24.2) to the generalized Ohm"s law can be done as follows. We multiply the third equation of (24.2) by $\vec{j}^*$ and integrate over the volume V of the conductors

$$\int_V \left(\frac{\vec{j}\,\vec{j}^*}{\sigma} - \vec{E}\,\vec{j}^* \right) dV = \int_V \vec{K}\,\vec{j}^*\, dV . \tag{25.1}$$

The density of Joule's heat is $q = \frac{\vec{j}\,\vec{j}^*}{2\sigma}$ [1], so that in the network

1. The evaluations are carried out for discrete harmonic oscillations of frequency ω, as a result of which the coefficients in the bilinear

resistance R corresponding to the energy determination we have

$$\int_V \frac{\vec{j}\,\vec{j}^*}{\sigma}\,dV = 2\int_V q\,dV = R I I^* \qquad (25.2)$$

where

$$I = \int_S \vec{j}\,dS \qquad (25.3)$$

is the total current strength (the same in all cross sections S, since $\mathrm{div}\,\vec{j} = 0$).

Let us now find the total magnetic energy; for this we shall integrate $\mu \vec{H}\,\vec{H}^*$ over the entire space. Using the first two equations of (24.2), we get

$$\int \mu \vec{H}\,\vec{H}^*\,dV = -\frac{1}{ik}\int \vec{H}^*\,\mathrm{curl}\,\vec{E}\,dV = -\frac{1}{ik}\int \left\{ \vec{E}\,\mathrm{curl}\,\vec{H}^* + \mathrm{div}\,[\vec{E}, \vec{H}^*] \right\} dV =$$

$$= -\frac{4\pi}{ikc}\int \vec{E}\,\vec{j}^*\,dV - \frac{1}{ik}\oint_\Sigma [\vec{E}, \vec{H}^*]\,\vec{N}\,d\Sigma = -\frac{4\pi}{ikc}\int_V \vec{E}\,\vec{j}^*\,dV. \qquad (25.4)$$

We took into account here that the space integral extends only over the conductor volume V, but the surface integral over the infinitely distant surface Σ equal, for the quasi-stationary plane, to zero. Surfaces of conductors are not special since the tangential components of $\vec{E}$ and $\vec{H}$ on them are continuous.

In accordance with the energy determination of the total induction L we have

$$U_m = \frac{1}{16\pi}\int \mu \vec{H}\,\vec{H}^*\,dV = \frac{L I I^*}{4c^2}. \qquad (25.5)$$

From (25.4) and (25.5) it follows that

$$\int_V \vec{E}\,\vec{j}^*\,dV = -\frac{i\omega L}{c^2} I I^*. \qquad (25.6)$$

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expressions are four times smaller than in the corresponding formulae for overall averaged spectral intensities.

Substituting (25.2) and (25.6) into (25.1) and using expression (25.3) for I^*, we obtain

$$I\left(R + \frac{i\omega L}{c^2}\right) = \frac{\int\limits_V \vec{K}\,\vec{j}^*\,dV}{\int\limits_S \vec{j}^*\,dS} . \qquad (25.7)$$

Whence it follows that the right hand side is the integral emf at frequency ω. As had been said, this emf as well as the total network impedance

$$Z = R + \frac{i\omega L}{c^2} \qquad (25.8)$$

depend on the distribution of the lateral field $\vec{K}$ in the conductor volume directly as well as on the distribution of $\vec{j}^*$ as determined by $\vec{K}$. We are however interested in another determination of the emf in which the lateral field $\vec{K}$ would enter explicitly. Such a determination can be obtained with the help of the reciprocity theorem.

Let us limit ourselves now to the case of a closed unramified network, consisting of quasi-linear conductors. Let us separate a small segment A (Fig. 23) whose impedance is negligibly small compared to the impedance of the whole network. Along with the case in which we are interested, when the lateral field $\vec{K}$ is distributed in the volume of all conductors and equations (24.2) apply, we shall study another, auxiliary case when the lateral field $\vec{K}_0$ is constant over the cross-section and acts only in the segment A. We shall denote all magnitudes corresponding to these conditions with the index 0:

$$\begin{aligned} \operatorname{curl} \vec{E}_0 &= -ik\mu \vec{H}_0 \\ \operatorname{curl} \vec{H}_0 &= \frac{4\pi}{c} \vec{j}_0 \qquad (25.9) \\ \vec{j}_0 &= \sigma(\vec{E}_0 + \vec{K}_0) . \end{aligned}$$

From the first of the equations (24.2) and (25.9) it follows that

$$\vec{H}_0 \operatorname{curl} \vec{E} - \vec{H} \operatorname{curl} \vec{E}_0 = 0 . \qquad (25.10)$$

On the other hand, using the second and third of the equations (24.2) and (25.9), we get

$$\vec{H}_0 \operatorname{curl} \vec{E} - \vec{H} \operatorname{curl} \vec{E}_0 = \vec{E} \operatorname{curl} \vec{H}_0 + \operatorname{div} [\vec{E}, \vec{H}_0] - \vec{E}_0 \operatorname{curl} \vec{H} -$$

$$- \operatorname{div} [\vec{E}_0, \vec{H}] = \frac{4\pi}{c} (\vec{E}\vec{j}_{\omega} - \vec{E}_0\vec{j}) + \operatorname{div} \left\{ [\vec{E}, \vec{H}_0] - [\vec{E}_0, \vec{H}] \right\} =$$

$$= \frac{4\pi}{c} (\vec{E}_0\vec{j} - \vec{E}\vec{j}_{\omega}) + \operatorname{div} \left\{ [\vec{E}, \vec{H}_0] - [\vec{E}_0, \vec{H}] \right\} .$$

Integrating this over the entire field and noting that the integral from the first term extends only over the volume V of the conductors and that the surface integral vanishes (the surfaces of the conductors need not be considered in view of the continuity of the tangential components on them), we obtain, in view of (25.1),

$$\int_V \vec{E}_0 \vec{j} \, dV = \int_V \vec{E} \vec{j}_0 \, dV . \qquad (25.11)$$

Let us denote by $\vec{s}$ the unit vector along the axis of the quasi-linear conductor and by ds an element of length along the axis. Since $\vec{j} = \vec{s}j$, $\vec{j}_0 = \vec{s}j_0$, $dV = ds dS$ and the field $\vec{E}_0$ is constant over the section, (25.11) takes the form

$$\int_V \vec{E}_0 \vec{j} \, dV = \int_A E_{0s} \, ds \int_S j dS = \mathcal{E}_0 I = \int_V E_s j_0 \, dV .$$

But $\mathcal{E}_0 = Z_0 I_0$, where

$$I_0 = \int_S j_0 dS, \qquad Z_0 = R_0 + \frac{i\omega L_0}{c^2}$$

(Z_0 is the network impedance with "empty" wires, i.e., not containing a lateral field, calculable with the help of j_0). Therefore,

$$IZ_0 = \frac{\int_V K_s j_0 dV}{\int_S j_0 dS} = \mathcal{E} . \qquad (25.12)$$

The determination of the equivalent integral emf, giving (25.12), is of the type which we had wanted to obtain: $\mathcal{E}$ depends on $\vec{K}$ only explicitly. This emf $\mathcal{E}$ permits us to find the correct value of the current I if we use the wire impedance Z_0 which does not contain lateral fields.

At very low frequencies, when j_0 can be considered constant over the wire section (absence of skin-effect), we have $\int_S j_0 dS = j_0 S = I_0 = \text{const.}$ and $\mathcal{E}$ takes the form

$$\mathcal{E} = \oint_\Gamma ds \frac{1}{S} \int_S K_s dS \qquad (25.13)$$

where Γ is the contour of the axis line of the network. In the case of a wire of constant cross-section, (25.13) gives

$$\mathcal{E} = \frac{1}{S} \int_S dS \oint_\Gamma K_s \, ds \qquad (25.14)$$

i.e., $\mathcal{E}$ represents in this case value of the circulation of $\vec{K}$ averaged over the cross-section.

If K_s is constant over the wire section, which is the case, for example, for induction in the external field, then (25.12) becomes the usual expression for the emf

$$\mathcal{E} = \oint_\Gamma K_s ds, \qquad (25.15)$$

The formula for $\mathcal{E}$, utilized in the previously mentioned work of M. A. Leontovich and of the author, refers to the special case of a round wire of constant cross-section. In cylindrical coordinates r, φ, z, since $j_0 \sim J_0(\varkappa r)$, where J_0 is the Bessel function of zero order, and $\varkappa^2 = -2i/d^2$, we then have

$$\mathcal{E} = \frac{\int_V K_s(r, \varphi, z)\, J_0(\varkappa r)\, dV}{\int_S J_0(\varkappa r)\, dS}$$

Section 26. Nyquist's Formula

Having formula (25.12), which gives $\mathcal{E}$ in terms of the strength of the lateral field $\vec{K}$ and having the correlation function for the s-component of $\overline{K}$

$$\overline{K_s(\vec{r}_1)K_s^*(\vec{r}_2)} = \frac{\theta}{\pi\sigma}\delta(\vec{r}_1 - \vec{r}_2) \qquad (26.1)$$

it is not difficult to obtain the spectral intensity of $\mathcal{E}$. According to (2.9), the spectral intensity decomposed with respect to positive frequencies ω is

$$\mathcal{E}_\omega^2 = 2\overline{|\mathcal{E}|^2}.$$

Substituting in here (25.12), we find

$$\mathcal{E}_\omega^2 = \frac{2}{I_0 I_0^*} \iint_{VV} \overline{K_s(\vec{r}_1) K_s^*(\vec{r}_2)}\, j_0(\vec{r}_1)\, j_0^*(\vec{r}_2)\, dV_1\, dV_2 .$$

Using then (26.1), we obtain

$$\mathcal{E}_\omega^2 = \frac{2\theta}{\pi I_0 I_0^*} \int_V \frac{j_0 j_0^*}{\sigma}\, dV .$$

But, according to (25.2)

$$\int_V \frac{j_0 j_0^*}{\sigma}\, dV = R_0 I_0 I_0^*$$

which finally gives Nyquist's formula

$$\mathcal{E}_\omega^2 = \frac{2\theta}{\pi} R_0 . \qquad (26.2)$$

In some cases, when we are talking of waves propagating in the wires and the formulation of the problem admits the introduction of distributed quasi-stationary parameters (the use of the telegraph equation), the linear density of $\mathcal{E}$ can be of advantage, i.e., emf per unit length of wire

$$\mathcal{E}' = \frac{\partial \mathcal{E}}{\partial s} . \tag{26.3}$$

The correlation function for $\mathcal{E}'$ will then be as follows

$$\overline{\mathcal{E}'(s_1)\,\mathcal{E}'^*(s_2)} = \frac{\theta}{\pi\sigma'}\,\delta(s_1 - s_2) \tag{26.4}$$

where σ' is the conductivity of unit length of "empty" (in the sense given above) wire. For the spectral intensity of the total emf in a wire of length ℓ, (26.2) follows from (26.3) and (26.4)

$$\mathcal{E}_\omega^2 = 2\overline{|\mathcal{E}|^2} = 2\int_0^\ell\int_0^\ell \overline{\mathcal{E}'(s_1)\,\mathcal{E}'^*(s_2)}\,ds_1\,ds_2 = \frac{2\theta}{\pi}\int_0^\ell \frac{ds}{\sigma'} = \frac{2\theta}{\pi}R_0 .$$

As is known, formula (26.2) is not restricted to the special assumptions made during its derivation. Derivations of $\mathcal{E}_\omega^2$ based on general premises of thermodynamics and statistics[1] give (26.2) independently of why and how the active resistance depends on ω (ramification of the network made of conductors of any section, presence of capacity, etc.). The skin-effect, to the extent that it influences the magnitude of the active resistance, is taken into account automatically.

Furthermore, our derivation refers to the case of good conductors, within which current perturbation can be neglected, i.e., (24.1) takes place, and for which, too, magnetic losses are absent (μ real, lateral magnetic field $H = 0$). From general thermodynamic considerations it is

1. H. Nyquist, Phys. Rev. 32, 110, 1928; J. Bernamont, Ann. de phys. 7, 71, 1937; C. J. Bakker and C. Heller, Physica 6, 262, 1939; S. S. Solomon, J. appl. Phys. 23, 109, 1952. For derivation in quantum domain, see H. B. Callen and Th. A. Welton, Phys. Rev. 83, 34, 1951, and also V. L. Ginsburg, U.F.N. 46, 398, 1952.

clear that both assumptions are not essential. In fact, basing ourselves on electrodynamic equations (1.3) or (1.9), which refer to the general case of complex ε and μ, and on the expressions for the correlation functions of $\vec{K}$ and $\vec{M}$ (section 9), we can show that formula (26.2) is valid for any conductor parameters or, in better words, for the "noisy" pattern, since the latter may be close to both the dielectric and the magnetic and can control losses of any origin. Understandably, the expression itself for the emf of $\mathcal{E}^{c}$ must in a corresponding manner be generalized in order to include also the lateral field $\vec{M}$.

The dependence on frequency of $\mathcal{E}_{\omega}^{2}$ can be governed not only by the parameters and the arrangement of the network, but also by the dependence on frequency of the permittivities ε and μ, and therefore by the correlation constants. For instance, in the case of good conductors, when $C = \frac{c}{\pi\sigma}$, the unit conductivity σ may depend on the frequency. Taking into account of the finite time of free travel of an electron in metal $\tau = s/v$ (s is the free path length, v is the thermal velocity) leads to a dispersion of σ, as a result of which the fluctuating current spectrum is cut off at frequencies $\omega \sim 1/\tau$.[1] In practice this effect in which a time correlation fluctuation of the current appears for time intervals less than τ does not play any role, since under ordinary conditions $s \sim 10^{-6}$ cm, $v \sim 10^{8}$ cm/sec for almost all metals, and, therefore, $1/\tau \sim 10^{14}$ sec^{-1}. It should be noted, however, that these frequencies are precisely of such an order that the "classical" skin-layer (∂ ω) becomes comparable to the free path length ($d \sim s$). Meanwhile, the authors of the work mentioned above consider electrons as particles reacting with a grid (in momentary acts of collision with ions), and do not take into account their magnetic field. As a result, the skin-effect drops out of the picture.[2]

1. C. J. Bakker and G. Heller, Physica 6, 262, 1939.

2. At low (helium) temperatures s can already in the region of radio frequencies ($\omega > 10^{10}$) exceed considerably the thickness of the so-called skin-layer of electrodynamics. See V. L. Ginzburg, UFN 42, 333, 1950; A. A. Abrikosov, DAN 86, 43, 1952; C. J. Grebenkemper and

Nyquist's formula is also valid, of course, for ultra-conductors. As a result of low temperatures and of small magnitudes of losses, the "noise" level must be very low,[1] which was recently confirmed by straight experiment.[2] With the precision used in the experiments (at frequency 5.6 mc, band 10 kc), it was not possible to uncover the "noise" of the ultra-conductor. It is of interest that "noise" is absent even then when a very strong current (about 100 amperes) is circulating in the ultra-conducting ring.

A term of the potential type can, in the general case, be present in the correlation function component of $\hat{K}$. According to (3.3) we have for the longitudinal (along the wire axis z) component of $\vec{K}$

$$F_{zz}(r) = \overline{K_z(\vec{r}_1)\, K_z^*(\vec{r}_2)} = \Phi(r) + \frac{\partial^2 \Psi(r)}{\partial z^2} \qquad (26.5)$$

where $r = |\vec{r}_1 - \vec{r}_2|$, $z = z_1 - z_2$. This expression is valid, however, only for points removed from the conductor surface by a distance greater than the correlation radius. In the surface layer of thickness of the order of a, F_{zz} will be anisotropic and, in the general case, nothing can be said about it.[3]

If the conductor is closed, the $\Phi(r)$ and $\Psi(z)$ will be periodic even functions of $z_1 - z_2$, with period equal to the length ℓ of the axial line of the conductor. In this case the potential type term does not give any contribution in the expression for $\mathcal{E}_\omega^2$ and can therefore be thrown out. If, however, the conductor is opened, then, generally speaking, the appearance of additional terms in $\mathcal{E}_\omega^2$ in (26.2) is possible

J. P. Hagen, Phys. Rev. 86, 673, 1952; A. B. Pippard and R. G. Chambers, Proc. phys. Soc. (A) 65, 955, 1952.

1. See V. L. Ginzburg, UFN 46, 348, 1952.

2. K. S. Knol and J. Volger, Physica 19, 46, 1953.

3. See M. A. Leontovich, DAN 53, 115, 1946 concerning the influence of the surface on the form of the correlation function in the concrete problem of the density fluctuations of a charge in electrolytic solutions.

due to the potential term in (26.5), if we consider this formula to remain valid right up to the butt end of the conductor, and due, as well, to the factual difference of F_{zz} from (26.5) in the surface layer. However, the additional terms do not depend on the length $\mathcal{L}$ of the conductor and can, therefore, give a correction to $\mathcal{E}_{\omega}^2$ to an order not lower than $a/\mathcal{L}$. Looking upon the ends of the conductor as condensers, we would then obtain a correction of the same order too for the total electric energy of this condenser

$$U = \int_0^\infty U_\omega d\omega = \frac{1}{2C}\int_0^\infty q_\omega^2 d\omega , \qquad \text{where} \qquad q_\omega^2 = 2\overline{|q|^2} = \mathcal{E}_\omega^2/\omega^2|Z|^2 .$$

The Nyquist emf (26.2) leads, as is known, to the value $U = \theta/2$.[1] The correction term of the order $a/\mathcal{L}$ does not contradict macroscopic thermodynamics, since a is a microquantity, but the question about the nature of such a correction term requires a separate investigation.

Section 27. Integral emf and Radiation

The concept of integral fluctuation emf can be utilized also in questions related to thermal radiation, provided all the dimensions of the emitting network are small compared to the wavelength in surrounding space. We shall study in this section only one example: we shall show how Nyquist's formula can be obtained from the condition of thermal equilibrium between an elementary helix (magnetic dipole) and the emission. The idea of the derivation belongs to M. A. Leontovich.[2]

Let us surround the elementary coil with a closed surface Σ. According to Poynting's theorem, the spectral intensity of energy flow through this

1. For R independent of ω. See G. C. Gorelik, UFN 44, 33, 1951; V. L. Ginsburg, UFN 46, 398, 1952.

2. In J. Slater's book "Ultra-short-wave Transfer" (M., 1946), in section 34, are contained some not very consequential discussions concerning thermal equilibrium in antenna systems, leaving the question of ohmic resistance and radiation resistance unclear (p. 288).

surface is

$$P_{\omega} = \oint_{\Sigma} S_{\omega N}\, d\Sigma = -\int_V (\overline{\vec{j}\,\vec{E}^*} + \overline{\vec{j}^*\vec{E}})\, dV .$$

The volume integral extends only over the region $\vec{j} \neq 0$, i.e., over the volume of wire of which the coil is made. Since this is a quasi-linear wire, we can write

$$P_{\omega} = -(I\mathcal{E}^* + I^*\mathcal{E}) \tag{27.1}$$

where $\mathcal{E} = \oint_{\Gamma} \vec{E}\, d\vec{s}$ is the emf, caused in the coil by the field $\vec{E}$.

Let the coil be placed in a given equilibrium (and, therefore, isotropic) field $\vec{E}_0$, created by exterior sources. Then the field $\vec{E}$ in the presence of the coil is composed of $\vec{E}_0$ and the field $\vec{E}_I$, created by the current I coursing in the coil. Since the presence of the coil must not disturb thermal equilibrium, the summed field $\vec{E}$ must also be an equilibrium field and, therefore, isotropic. Therefore the average energy flow P_{ω} through the closed surface Σ must equal zero

$$\overline{I\mathcal{E}^*} + \overline{I^*\mathcal{E}} = 0 . \tag{27.2}$$

In accordance with the decomposition of $\vec{E}$ into $\vec{E}_0$ and $\vec{E}_I$, let us similarly decompose the total emf $\mathcal{E}$ into the emf $\mathcal{E}_0$, caused in the coil by the field of external sources, and the emf $\mathcal{E}_I$, which determines in the coil the presence of resistance of radiation ρ and of the inductance L

$$\mathcal{E} = \mathcal{E}_0 + \mathcal{E}_I ; \quad \mathcal{E}_I = -\left(\rho + \frac{i\omega L}{c^2}\right) I . \tag{27.3}$$

Substituting (27.3) into (27.2), we obtain

$$\overline{I\mathcal{E}_0^*} + \overline{I^*\mathcal{E}_0} = 2\rho\overline{|I|^2} . \tag{27.4}$$

This equation means that the work of the external emf $\mathcal{E}_0$ covers the losses of radiation of the coil.

If R is the ohmic resistance of the coil at frequency ω, then the generalization of Ohm's law can be written in the form

$$RI = \mathcal{E} + \mathcal{E}_{fl.} \tag{27.5}$$

wherein is introduced some fluctuation emf $\mathcal{E}_{fl.}$, which must be picked up in such a way that the radiation of the coil does not disturb equilibrium, i.e., that (27.2) or correspondingly (27.4) take place. Carrying (27.3) into (27.5), we obtain

$$ZI = \mathcal{E}_0 + \mathcal{E}_{fl.} \tag{27.6}$$

where

$$Z = R + \rho + \frac{i\omega L}{c^2} \tag{27.7}$$

is the impedance of the coil.

Substituting the current I from (27.6) into (27.4) and taking into account that the external emf $\mathcal{E}_0$ and the "local" fluctuation emf $\mathcal{E}_{fl.}$ are not correlated between themselves, i.e.,

$$\overline{\mathcal{E}_0 \mathcal{E}^*_{fl.}} = \overline{\mathcal{E}^*_0 \mathcal{E}_{fl.}} = 0 \tag{27.8}$$

we find

$$\overline{|\mathcal{E}_{fl.}|^2} = \frac{R}{\rho} \overline{|\mathcal{E}_0|^2}. \tag{27.9}$$

Let us now study $\overline{|\mathcal{E}_0|^2}$. Let H_{0n} denote the component normal to the coil surface of the magnetic field of external sources. Then $\mathcal{E}_0 = -\frac{i\omega}{c} SH_{0n}$, where S is the coil area. Consequently

$$\overline{|\mathcal{E}_0|^2} = \frac{\omega^2 S^2}{c^2} \overline{|H_{0n}|^2}. \tag{27.10}$$

But, according to condition, the radiation of external sources is isotropic, so that

$$\overline{|H_{0x}|^2} = \frac{1}{6}\left\{\overline{|E_0|^2} + \overline{|H_0|^2}\right\} = \frac{2\pi}{3}u_{0\omega} = \frac{2\theta\omega^2}{3\pi c^3} \qquad (27.11)$$

where we have used the Rayleigh-Jeans law (5.4):

$$u_{0\omega} = \frac{1}{4\pi}\left\{\overline{|E_0|^2} + \overline{|H_0|^2}\right\} = \frac{\theta\omega^2}{\pi^2 c^3} .$$

From (27.10) and (27.11), introducing the resistance of radiation of the elementary coil $\rho = \frac{2\omega^4 S^2}{3c^5}$, [1] we obtain

$$\overline{|\mathcal{E}_0|^2} = \frac{\omega^2 S^2}{c^2}\cdot\frac{2\theta\omega^2}{3\pi c^3} = \frac{\theta}{\pi}\rho . \qquad (27.12)$$

According to (27.9) and (27.12), the spectral intensity, introduced into Ohm's law, of the fluctuation emf is

$$\mathcal{E}^2_{fl.\omega} = 2\overline{|\mathcal{E}_{fl.}|^2} = \frac{2R}{\rho}\cdot\frac{\theta}{\pi}\rho = \frac{2\theta}{\pi}R . \qquad (27.13)$$

Thus, for $\mathcal{E}_{fl.}$ we come to Nyquist's formula. Understandably, this derivation can be reversed and, considering Nyquist's formula to be known, the Rayleigh-Jeans law can be obtained from (27.9) and (27.11).

Multiplying (27.5) by I^*, taking the real part and utilizing (27.2), it is not difficult to convince oneself that

$$\overline{I\mathcal{E}^*_{fl.}} + \overline{I^*\mathcal{E}_{fl.}} = 2R\,\overline{|I|^2} \qquad (27.14)$$

i.e., the work of the fluctuation emf $\mathcal{E}_{fl.}$ is spent on perturbing Joule losses in the coil.

1. J. A. Stratton, "Theory of Electromagnetism," Section 8.6 (M., 1948).

The derivation given relies on the confirmation in (27.8) of the absence of space correlation between the emf $\mathcal{E}_0$, determined by the field of external sources, and $\mathcal{E}_{fl.}$. If we imagine that the coil is surrounded by a closed conducting envelope and that the entire system is at thermal equilibrium, then it is not difficult to establish that (27.8), generally speaking, does not take place. In fact, the currents in the envelope are partially conditioned by the field $\vec{E}_I$ of the coil current, and this current is partly correlated to $\mathcal{E}_{fl.}$. Thereby, a correlation between $\mathcal{E}_0$ and $\mathcal{E}_{fl.}$ exists. However, as is clear from general thermodynamic reasons, the result (27.13) must not be related with the electrodynamic parameters of external bodies, since we are talking about an equilibrium condition. We can therefore assume, that external bodies (or the envelope in the adduced example) are absolutely black. In this case it is evident without further ado that the field $\vec{E}_0$ created by the envelope, since it does not reflect radiation from the coil, can be considered fixed and that the emf $\mathcal{E}_0$ is not correlated with $\mathcal{E}_{fl.}$.

Let us now suppose that the coil emits into free space, i.e., the non-equilibrium state is studied for which (27.8) does not take place. The field $\vec{E}_0$ and the emf $\mathcal{E}_0$ are equal in this case to zero and Ohm's law (27.6) takes the form

$$ZI = \mathcal{E}_{fl.}$$

whence

$$\overline{I\mathcal{E}^*_{fl.}} + \overline{I^*\mathcal{E}_{fl.}} = 2(R+\rho)\,\overline{|I|^2}. \qquad (27.15)$$

Thus, the work of the fluctuation emf covers the losses for the Joule heat as well as radiation. Using Nyquist's formula (27.13), we find that the power equals

$$P_\omega = 2\rho|I|^2 = 2\rho\frac{|\mathcal{E}_{fl.}|^2}{|Z|^2} = \frac{2\theta\rho R}{\pi\left\{(\rho+R)^2 + \frac{\omega^2 L^2}{c^4}\right\}}. \qquad (27.16)$$

Multiplying numerator and denominator by $|I|^4/4$, where I is the harmonic current of frequency ω and of some arbitrary amplitude, and taking (25.2) and (25.5) into account, (27.16) can be expressed in terms of energy magnitudes -- the power $P = \frac{\rho|I|^2}{2}$ emitted by current I, the corresponding Joule heat $Q = \frac{R|I|^2}{2}$ and the energy of the quasi-stationary magnetic field $U_m = \frac{L|I|^2}{4c^2}$:

$$P_\omega = \frac{2\theta}{\pi} \frac{PQ}{(P+Q)^2 + 4\omega^2 U_m^2}. \qquad (27.17)$$

This form of writing is more convenient in the cases when we talk of radiation from systems small compared to the wavelength but factually are not an elementary coil, for instance, radiation from a small well-conducting sphere. The evaluation of P, Q and U_m for this latter case (see Appendix IX) brings (27.17) into the form

$$P_\omega = \frac{3\theta}{\pi} \frac{\delta\alpha^2}{\left(\alpha^3 + \frac{\delta}{2\alpha}\right)^2 + 1} \quad (\delta = k\mu d, \quad \alpha = ka). \quad (27.18)$$

Since $\delta \ll \alpha \ll 1$, we can neglect the first term in the denominator (the losses of magnetic energy are considerably greater for a period) and then

$$P_\omega = \frac{3\theta\delta\alpha^2}{\pi}.$$

It is to this expression that the straight evaluation of P_ω in section 14 for a small sphere [formula (14.12)] had brought us.

CONCLUSIONS

The theory of electric fluctuations presented here consists of the unification of macroscopic electrodynamics, which describes electric and magnetic properties of bodies with the help of permittivities ε and μ, with space-correlation statistics, which relies on the representation of lateral fields introducible into the generalized Ohm's law in one or the other of its forms.

The electrodynamic side of the theory is thus general and methodically regular to the same extent as is the usual theory of the macroscopic field. The particularity of fluctuation boundary problems consists of the fact that one must always deal with either non-homogeneous differential equations or with non-homogeneous boundary conditions depending on whether the volume or the surface lateral field is utilized. The problems of the second type are considerably simpler, since it is not necessary to examine the field within radiating bodies in such problems. This simplification is admissible in the presence of a sufficiently well-developed skin-effect and is achieved by means of the generalized approximate boundary conditions of M. A. Leontovich (sections 11-13).

The statistical part of the problem is in many cases taken care of by the assumption of a space δ-correlation of the lateral fields. This assumption is sufficient for problems concerning the radiation of bodies into external space and for problems concerning the fluctuation integral electric magnitudes in quasi-stationary networks.

For the case of a transparent external medium (in particular -- vacuum) the thermal radiation intensity I_{ω} is everywhere finite, despite the δ-correlation of the lateral fluctuation field in the interior of the emitting body. This is a consequence of the fact that I_{ω} is related only to travelling waves. To I_{ω} corresponds everywhere the finite energy density $u_{\omega\,\text{waves}}$. But, beside the wave field, the emitting body also creates the quasi-stationary thermal field (non-uniform standing waves), localized at the surface of the body. In the transparent external medium the quasi-stationary field does not influence I_{ω}, but its energy density $u_{\omega\,\text{quas.}}$ depends on

a in the immediate vicinity of the surface -- at distances of the order of the correlation radius a of the lateral field -- and when a $\rightarrow$ 0 tends to infinity as $1/a^3$ (sections 6, 16).

In the case of an absorbing external medium, if the conditions of the problem permit neglect of its own thermal emission, it is similarly possible to find the energy flow, created by the emitting body, and to determine the corresponding intensity I_{ω}. The latter weakens as the distance from the surface increases not only because of geometric factors, but also as a result of absorption. Since, in the absorbing medium, there is not any sharp division between the travelling and non-uniform waves (in the creation of energy flow the entire field acts together as one), the indicated intensity can no longer be related in a simple way to the reflection coefficient (section 6).

A finite correlation radius is necessary not only for the determination of the energy of the external quasi-stationary field in volumes V adjacent to the body surface, but also for the finding of energy magnitudes within the absorbing medium. A definite meaning can be given to that part of these quantities, which refers to radiation and is independent of the correlation radius, only in the approximation into which it is at all possible to introduce the concept of radiation ability and intensity, i.e., with a precision up to the second order with respect to the coefficient of attenuation (sections 5, 8).

In the quasi-stationary region, the finite correlation radius of the lateral field in the case of open networks, generally speaking, can influence the spectral intensities of the fluctuating integral electric magnitudes, but the corresponding corrections -- if they exist -- are knowingly very small since they are related only to the surface layer of a thickness of the order of a (section 26).

The solution of fluctuation problems on the general electrodynamic basis permits very closely to express the asymptotic nature of the laws of classical radiation theory. If the cavity dimensions are not large compared to the wavelength λ and if the spectral interval $\Delta\lambda$ under consideration is not very broad, then the equilibrium radiation is not uniform

and isotropic. Under contrary conditions when interference effects are smoothed and approximations of geometric optics become applicable, asymptotic laws of classical radiation theory enter into force (sections 7, 16, 20).

For an isotropic transparent medium the passage to the asymptotic approximation ($\omega \rightarrow \infty$) brings us, in the case of a real refraction index, to the usual laws for intensity and density of equilibrium radiation energy

$$I_{\omega} = I_{0\omega} n^2, \qquad u_{\omega} = u_{0\omega} n^2 \left| \frac{\partial n\omega}{\partial \omega} \right| \quad (n^2 > 0)$$

(in the absence of scattering $u_{\omega} = u_{0\omega} n^3$). For an imaginary refraction index the same electrodynamic formulae give

$$I_{\omega} = 0, \qquad u_{\omega\,\mathrm{wave}} = 0, \qquad (n^2 < 0) \; ;$$

the emission of bodies in such a medium suffers complete reflection (section 7). The theory under consideration permits a new approach to the finding of laws which generalize the writing for the case of anisotropic and magnetoactive transparent media. As in the case of isotropic media, in the solution of the electrodynamic problem the passage to the asymptotic approximation must be made for sufficiently high frequencies, which leads to the desired expression for I_{ω} and u_{ω} (sections 20, 21, 23).

Of special interest is the question concerning thermal radiation in the case of bodies whose dimensions are comparable to λ and for which, therefore, the usual ("tri-fold") passage to the asymptotic approximation is not useful. Since the theory under study does not in any way limit the relation between body dimensions and wavelength, with its help one can study all possible problems of this nature concerning both emission into free space (sections 10, 14), and into any transfer lines (sections 15, 17). Diffraction phenomena are automatically included in this solution of the corresponding boundary problem.

There is a theorem concerning the spectral density of the total energy

flow which can be formulated as follows. Let us for brevity call "uniform" the union of waves, which propagate along any one coordinate in some, generally speaking, curved coordinate system [for instance, along the radius in the cylindrical (section 10) or spherical (section 14) coordinate systems, along one of the cartesian coordinates (section 15)]. Then, if the wave electromagnetic field in the space available to it and surrounding the emitting body can be represented in the form of a union of mutually non-interfering waves (mutually orthogonal) and if $A_{mn}(\omega)$ is the energy coefficient of attenuation by the given body of a single wave from this union, then the total power of thermal radiation of the body in the frequency interval $(\omega, \omega + d\omega)$ is

$$P_{\omega} = \frac{2\pi^2 I_{0\omega}}{k^2} \sum_{m,n} A_{mn}(\omega)$$

where summation extends over all values of m and n, corresponding to travelling waves. In waveguides the possibility of physically separating some of the partial waves exists; in the propagation in free space, however, the entire sum is of direct interest. The sum gives in this case dependence of P_{ω} on the electric parameters of the radiating body as well as on the relation between the body dimensions and the wavelength in the surrounding space.

Basing ourselves on the indicated uniform nature of the adduced theorem, we called it the "waveguide" form of Kirchhoff's law. This theorem does not take into account the interference phenomena in systems whose extent in the propagation direction is comparable to λ, and, in this sense, is asymptotic. In addition, the theorem concerns only travelling waves (in waveguides -- subcritical waves). These are precisely those conditions under which a classical approach is possible, an approach which relies, on the one hand, on the asymptotic behavior of the natural values (in the given case -- transverse oscillations of a homogeneous continuum) and, on the other hand on the theorem of energy partition according to degrees of freedom. We utilized this classical method of proof, firstly,

in relation to the waveguide form of Kirchhoff's law (section 17) and, secondly, in the derivation of the asymptotic expression for equilibrium radiation energy in anisotropic and magnetic media (section 22).

As had been already mentioned in the introduction, the theory studied leaves aside fluctuation problems, in which the parameters, characterizing the electrical microstructure of bodies and for which therefore the application of statistical electronics[1] is essential, play an important role. But the wide circle of questions -- from asymptotic ("optical") laws of the classical radiation theory to electric "noises" in quasi-stationary networks -- is grasped and illuminated from a single point of view. Understandably, the results presented in this work do not exhaust this circle of questions.

We are talking not only of the application of the described methods to different kinds of concrete problems, but also of some evidently begging generalizations. To their number belongs, for instance, the question mentioned in section 16 concerning the form of the correlation function of lateral fields in the case of anisotropic absorbing media.

Furthermore, in the course of the whole work (with the small exception in section 17) we limited ourselves to the examination of homogeneous and uniformly heated bodies. The question of thermal radiation in non-homogeneous media had already been studied earlier.[2] If the non-homogeneities are such that the approximations of geometric optics (sufficiently smooth variation in the properties of the medium) are still valid, then we need not expect anything new from the electrodynamic theory of fluctuation compared to what the classical theory of radiation can give (to this can be referred, in particular, the questions of radio-emission from non-earthly sources, among these the sun spots). But the general

1. If the result of the electro-statistical investigation can be expressed in terms of correlation functions for some lateral field, then all further questions are reduced to the scheme already investigated. This remark concerns electric fluctuations of any (not necessarily thermal) origin.

2. See M. Born and R. Ladenburg, Phys. Zet., 12, 198, 1911; W. Voigt, Wied. Ann. 35, 1381, 1912.

electrodynamic basis of the theory permits -- at least, in principle -- extension of the theory to the case when the electric parameters and the temperature vary appreciably with distances, comparable to the wavelength. Conditions of this nature can be encountered in micro-wave radio technology.

APPENDIX

I. Derivation of Formula (6.13)

Rewrite equations (6.9) and (6.10) into components and replace a and b according to formula (6.4), and get

$$u_1 - v_1 = \frac{\varepsilon - 1}{\varepsilon} \int_{-\infty}^{+\infty} \frac{dp_3}{p^2 - s^2} \left[(p_1^2 - s^2) g_1 + p_1 p_2 g_2 \right]$$

$$u_2 - v_2 = \frac{\varepsilon - 1}{\varepsilon} \int_{-\infty}^{+\infty} \frac{dp_3}{p^2 - s^2} \left[p_1 p_2 g_1 + (p_2^2 - s^2) g_2 \right]$$

$$p_1 u_3 - s_3 u_1 - \mu (p_1 v_3 - t_3 v_1) = - \frac{\varepsilon - 1}{\varepsilon} p_1 s^2 \int_{-\infty}^{+\infty} \frac{g_3 dp_3}{p^2 - s^2} \tag{I.1}$$

$$p_2 u_3 - s_3 u_2 - \mu (p_2 v_3 - t_3 v_2) = - \frac{\varepsilon - 1}{\varepsilon} p_2 s^2 \int_{-\infty}^{+\infty} \frac{g_3 dp_3}{p^2 - s^2}$$

$$p_1 u_1 + p_2 u_2 + s_3 u_3 = 0, \qquad p_1 v_1 + p_2 v_2 + t_3 v_3 = 0 .$$

Here, equalities (6.2), (6.6) and (6.8) have been used. Eliminating u and taking into account that $s^2 = k^2 \varepsilon \mu$ and $p^2 - s^2 = p_3^2 - s_3^2$, we get

$$v_1 s_3 \alpha + v_3 p_1 \beta = - k^2 \mu (\varepsilon - 1) s_3 \int_{-\infty}^{+\infty} \frac{(p_1 g_3 + s_3 g_1) dp_3}{p_3^2 - s_3^2}$$

$$v_2 s_3 \alpha + v_3 p_2 \beta = - k^2 \mu (\varepsilon - 1) s_3 \int_{-\infty}^{+\infty} \frac{(p_2 g_3 + s_3 g_2) dp_3}{p_3^2 - s_3^2} \tag{I.2}$$

$$p_1 v_1 + p_2 v_2 + t_3 v_3 = 0$$

where the following notations have been introduced:

$$\alpha = \mu t_3 - s_3, \qquad \beta = t_3 - \mu s_3 . \tag{I.3}$$

Let us also introduce

$$L = (p_1^2 + p_2^2)\beta - s_3 t_3 \alpha = t_3 s^2 - \mu s_3 t^2 = k^2 \mu(\varepsilon t_3 - s_3). \tag{I.4}$$

Then we get from (I.2)

$$\left.\begin{matrix} v_1 \\ v_2 \\ v_3 \end{matrix}\right\} = -\frac{k^2\mu(\varepsilon - 1)s_3}{\alpha\Delta}\int_{-\infty}^{+\infty}\frac{dp_3}{p_3^2 - s_3^2} \times$$

$$\times \begin{cases} [(p_2^2\beta - s_3t_3\alpha)g_1 - p_1p_2\beta g_2 - p_1t_3\alpha g_3] \\ [-p_1p_2\beta g_1 + (p_1^2\beta - s_3t_3\alpha)g_2 - p_2t_3\alpha g_3] \\ \alpha[s_3(p_1g_1 + p_2g_2) + (p_1^2 + p_2^2)g_3] . \end{cases} \tag{I.5}$$

In accordance with (6.7) and (6.11), we have

$$S_z = -\frac{c}{4\pi k}\iint_{-\infty}^{+\infty}\left\{\overline{[\vec v[\vec t^*, \vec v^*]]e^{i(\vec t - \vec t^*)z}} + c.c.\right\} dp_1' dp_2' dp_1'' dp_2'' =$$

$$= -\frac{c}{4\pi k}\iint_{-\infty}^{+\infty}\left\{(\vec t + \vec t^*)\overline{\vec v \vec v^*} - \overline{\vec v^*(\vec t^*, \vec v)} - \right.$$

$$\left. - \overline{\vec v(\vec t, \vec v^*)}\right\} e^{i(t_3 - t_3^*)z}\, dp_1'\, dp_2'\, dp_1''\, dp_2'' \tag{I.6}$$

where the conjugate quantities are functions of p_1'', p_2'' and the quantities without asterisks are functions of p_1', p_2' . Using the correlation function

(6.12), let us evaluate the quantities entering under the integral sign in (I.6). We have

$$\overline{\vec{v}\,\vec{v}^*} = \frac{k^4\mu^2|\varepsilon - 1|^2 C}{4\pi^3}\delta(p_1' - p_1'')\delta(p_2' - p_2'')\ \times$$

$$\times \left\{ \frac{1}{|\alpha|^2} + \frac{|t|^2|s|^2}{|\Delta|^2} \right\} |s_3|^2 \int_{-\infty}^{+\infty} \frac{dp_3}{|p_3'^2 - s_3^2|^2}\ , \tag{I.7}$$

$$\overline{(\vec{t}^*, \vec{v})} \begin{Bmatrix} v_1^* \\ v_2^* \\ v_3^* \end{Bmatrix} = \frac{k^4\mu^2|\varepsilon - 1|^2 C}{4\pi^3}\delta(p_1' - p_1'')\delta(p_2' - p_2'')\ \times$$

$$\times\ \frac{|s|^2|s_3|^2}{|\Delta|^2}(t_3 - t_3^*) \begin{Bmatrix} t_3^* p_1' \\ t_3^* p_2' \\ p_1'^2 + p_2'^2 \end{Bmatrix} \int_{-\infty}^{+\infty} \frac{dp_3'}{|p_3'^2 - s_3^2|^2}\ . \tag{I.8}$$

Since s_3 determines the change in amplitude of the reflected field along the coordinate z and as the distance from the boundary increases in the direction of negative z's this field must become dampened, the imaginary part of s_3 must be negative. From here it follows that the poles $p_3' = -s_3$ and $p_3' = s_3^*$ lie in the upper semi-plane of the complex variable p_3'. Making calculations at these poles, we obtain

$$\int_{-\infty}^{+\infty} \frac{dp_3'}{|p_3'^2 - s_3^2|^2} = \frac{i\pi}{|s_3|^2(s_3^* - s_3)}\ . \tag{I.9}$$

Introduce this value of the integral into expressions (I.7) and (I.8) and then introduce these expressions into (I.6). We obtain for the component S_ω the following result (we can now omit the primes on p_1' and p_2'):

$$S_{\omega j} = -\frac{k^3 c i \mu^2 |\varepsilon - 1|^2 C}{16\pi^3} \int_{-\infty}^{+\infty} \frac{p_j dp_1 dp_2}{s_3^* - s_3} e^{i(t_3 - t_3^*)z} \left\{ 2 \frac{1}{|\alpha|^2} + \frac{|t|^2 |s|^2}{|\Delta|^2} - \frac{(t_3 - t_3^*)^2 |s|^2}{|\Delta|^2} \right\} \quad (j = 1, 2) \tag{I.10}$$

$$S_{\omega 3} = -\frac{k^3 c i \mu^2 |\varepsilon - 1|^2 C}{16\pi^3} \int_{-\infty}^{+\infty} \frac{(t_3 + t_3^*) dp_1 dp_2}{s_3^* - s_3} e^{i(t_3 - t_3^*)z} \left(\frac{1}{|\alpha|^2} + \frac{|t|^2 |s|^2}{|\Delta|^2} \right).$$

Let us now note that t_3, s_3, α and Δ are even functions of p_1 and p_3. The components $S_{\omega 1}$ and $S_{\omega 2}$ therefore become zero (the expression under the integral sign is uneven in p_1 for the first one, and for second in p_2). Further,

$$t_3 = \sqrt{k^2 - (p_1^2 + p_2^2)}$$

can assume either a real or an imaginary value. Since in the latter case $t_3 + t_3^* = 0$, the limits of integration in $S_{\omega 3}$ narrow down to the domain $p_1^2 + p_2^2 \leq k^2$, i.e., in the domain of real t_3.

For real t_3's we have $|t|^2 = t^2 = k^2$, so that

$$S_{\omega 1} = S_{\omega 2} = 0$$

$$S_{\omega 3} = -\frac{k^3 c i \mu^2 |\varepsilon - 1|^2 C}{8\pi^3} \int_{p_1^2 + p_2^2 \leq k^2} \frac{t_3 dp_1 dp_2}{s_3^* - s_3} \left(\frac{1}{|\alpha|^2} + \frac{k^2 |s|^2}{|\Delta|^2} \right). \tag{I.11}$$

Let us now represent t in spherical coordinates with axis Z

$$t_1 = p_1 = k \sin\theta \cos\varphi, \quad t_2 = p_2 = k \sin\theta \sin\varphi,$$

$$t_3 = k\cos\theta . \tag{I.12}$$

Then

$$p_1^2 + p_2^2 = k^2\sin^2\theta, \qquad s_3 = k\sqrt{\epsilon\mu - \sin^2\theta} \equiv k\zeta, \tag{I.13}$$
$$dp_1\,dp_2 = k^2\cos\theta\sin\theta\,d\theta\,d\varphi = k^2\cos\theta\,d\Omega,$$

where integration with respect to the solid angle Ω extends over the hemisphere $\theta \leqslant \pi/2$. Finally, in accordance with (1.3) and (1.4),

$$\Delta_\mu = \mu t_3 - s_3 = -k(\mu\cos\theta + \zeta),$$
$$\Delta = k^2\mu(t_3\epsilon - s_3) = -k^3\mu(\epsilon\cos\theta + \zeta). \tag{I.14}$$

Finally we obtain

$$S_{\omega 3} = \frac{k^3 c\, i\, |\epsilon - 1|^2 C}{8\pi^3}\int_{\theta\leqslant\pi/2}\frac{\cos^2\theta\,d\Omega}{\zeta^* - \zeta}\times$$
$$\times\left(\frac{\mu^2}{|\mu\cos\theta + \zeta|^2} + \frac{\sin^2\theta + |\zeta|^2}{|\epsilon\cos\theta + \zeta|^2}\right)$$

Comparison of this formula with (5.7) gives the following expression for the intensity

$$I_\omega = \frac{k^3 c\, i\, |\epsilon - 1|^2 C}{8\pi^3}\,\frac{\cos^2\theta}{\zeta^* - \zeta}\left(\frac{\mu^2}{|\mu\cos\theta + \zeta|^2} + \frac{\sin^2\theta + |\zeta|^2}{|\epsilon\cos\theta + \zeta|^2}\right). \tag{I.15}$$

Taking into account that

$$\frac{\mu^2}{\zeta^* - \zeta} = \frac{\mu(\zeta^* + \zeta)}{\epsilon^* - \epsilon}, \qquad \frac{\sin^2\theta + |\zeta|^2}{\zeta^* - \zeta} = \frac{\epsilon\zeta^* + \epsilon^*\zeta}{\epsilon^* - \epsilon}$$

we can rewrite (I.15) in the form

$$I_{\omega} = \frac{k^3 c i |\varepsilon - 1|^2 c}{8\pi^3 i(\varepsilon^* - \varepsilon)} \left(\frac{\mu \cos\theta(\zeta^* + \zeta)}{|\mu\cos\theta + \zeta|^2} + \frac{\cos\theta(\varepsilon\zeta^* + \varepsilon^*\zeta)}{|\varepsilon\cos\theta + \zeta|^2} \right) ,$$

from which formula (6.13) follows.

If the half-space $z > 0$ is filled with a transparent medium with permittivities ε_1, μ_1, then the following changes must be introduced into the formulae in section 6 and in this Appendix. In the expression for $\vec{H}$ in (6.7) μ_1 will enter the denominator, and consequently in (I.1), (I.3) and (I.4) instead of μ we shall have μ/μ_1. The expression for t_3 will now assume the following form

$$t_3 = -\sqrt{k^2 \varepsilon_1 \mu_1 - (p_1^2 + p_2^2)} \qquad \text{(I.16)}$$

and instead of (I.14), we shall have

$$\alpha = -\frac{k}{\mu_1}\left(\mu \sqrt{\varepsilon_1 \mu_1} \cos\theta + \mu_1 \zeta_1\right)$$

$$\Delta = -k^3 \mu (\varepsilon \sqrt{\varepsilon_1 \mu_1} \cos\theta + \varepsilon_1 \zeta_1)$$

$$\zeta_1 = \sqrt{\varepsilon\mu - \varepsilon_1 \mu_1 \sin^2\theta} .$$

μ_1 will appear in the denominator in formula (I.11) for $S_{\omega 3}$, and the region of integration (region of actual t_3) will now be different.

If $\varepsilon_1 \mu_1 > 0$ then, according to (I.16), integration with respect to p_1 and p_2 must be extended to the circle $p_1^2 + p_2^2 < k^2 \varepsilon_1 \mu_1$. In this case, the following expression is obtained for the intensity I_{ω}

$$I_{\omega} = I_{c\omega} \varepsilon_1 \mu_1 (1 - R) = I_{0\omega} \varepsilon_1 \mu_1 \left(1 - \frac{R_{\perp} + R_{\parallel}}{2}\right) ,$$

where now

$$R_{\perp} = \left| \frac{\mu\sqrt{\varepsilon_1\mu_1}\cos\theta - \mu_1\zeta_1}{\mu\sqrt{\varepsilon_1\mu_1}\cos\theta + \mu_1\zeta_1} \right|^2 , \quad R_{\parallel} = \left| \frac{\varepsilon\sqrt{\varepsilon_1\mu_1}\cos\theta - \varepsilon_1\zeta_1}{\varepsilon\sqrt{\varepsilon_1\mu_1}\cos\theta + \varepsilon_1\zeta_1} \right|^2$$

If, however, $\varepsilon_1\mu_1 < 0$, then for any value of p_1 and p_2 we have imaginary values t_3, so that $t_3 + t_3^* = 0$ and, according to (I.10), $S_{\omega 3} = 0$. Consequently, in this case $I_{\omega} = 0$.

To evaluate the energy density in the half-space $z > 0$, one must substitute (6.7) into the expression

$$u_{\omega} = \frac{1}{4\pi}\left\{\overline{\vec{E}\vec{E}^*} + \overline{\vec{H}\vec{H}^*}\right\}$$

which gives

$$u_{\omega} = \frac{1}{4\pi}\iint_{-\infty}^{+\infty}\left\{\overline{\vec{v}\,\vec{v}^*}\left(1 + \frac{|t|^2}{k^2}\right) - \frac{\overline{(\vec{t},\vec{v}^*)(\vec{t}^*,\vec{v})}}{k^2}\right\} e^{i(t_3 - t_3^*)z}\, dp_1' dp_2' dp_1'' dp_2'' .$$

Using then formulae (I.7) - (I.9), substituting for C the value (6.15) (in which it is advantageous now to replace $I_{0\omega}$ by $u_{0\omega} = \frac{c}{4\pi} I_{0\omega}$) and transforming to polar coordinates

$$p_1 = p\cos\varphi, \quad p_2 = p\sin\varphi, \quad dp_1 dp_2 = 1 \quad \varphi,$$

we get

$$u_{\omega} = \frac{k\mu^2(\varepsilon^* - \varepsilon)}{4} u_{0\omega}\int_0^{\infty}\frac{p\,dp}{s_3^* - s_3}\left\{\left(\frac{1}{|\alpha|^2} + \frac{|t|^2|s|^2}{|\Delta|^2}\right) \times\right.$$

$$\left. \times \left(1 + \frac{|t|^2}{k^2}\right) + \frac{(t_3 - t_3^*)^2 p^2 |s|^2}{k^2|\Delta|^2}\right\} e^{i(t_3 - t_3^*)z}$$

where α and Δ are expressed by formulae (I.3) and (I.4).

Let us now divide the interval of integration with respect to p into two parts: from 0 to k (travelling waves, $t_3 = t_3^* = -\sqrt{k^2 - p^2}$, $|t|^2 = k^2$) and from k to ∞ (non-uniform standing waves, $t_3 = -t_3^* = i\sqrt{p^2 - k^2}$, $|t|^2 = 2p^2 - k^2$). Beside that let us take into account that, according to (6.6),

$$\frac{\mu^2}{s_3^* - s_3} = \frac{\mu(s_3^* + s_3)}{k^2(\varepsilon^* - \varepsilon)}, \qquad \frac{|s|^2}{s_3^* - s_3} = \frac{\varepsilon^* s_3 + \varepsilon s_3^*}{\varepsilon^* - \varepsilon}.$$

As a result, u_ω takes the form

$$\left.\begin{aligned} u_\omega = \frac{u_{0\omega}}{2k}\int_0^k \left\{ \frac{\mu(s_3^* + s_3)}{|\mu t_3 - s_3|^2} + \frac{\varepsilon^* s_3 + \varepsilon s_3^*}{|\varepsilon t_3 - s_3|^2} \right\} p\,dp \;+ \\ + \frac{u_{0\omega}}{2k}\int_k^\infty \left\{ \frac{\mu(s_3^* + s_3)}{|\mu t_3 - s_3|^2} + \frac{\varepsilon^* s_3 + \varepsilon s_3^*}{|\varepsilon t_3 - s_3|^2} \right\} e^{-2\sqrt{p^2 - k^2}\,z}\, p^3 dp\;. \end{aligned}\right\} \tag{I.17}$$

The first term gives the energy density of radiation $u_{\omega\,\text{waves}}$, independent of z, the second the decreasing with increasing z energy density $u_{\omega\,\text{quas}}$ of the quasi-stationary field. Introducing the change in variable ξ

$$p = k\sqrt{1 + \xi^2}, \quad t_3 = ik\xi, \quad s_3 = -ik\zeta, \quad \zeta = \sqrt{\xi^2 + 1 - \varepsilon\mu},$$

we transform the second term to the form

$$u_{\omega\,\text{quas}} = \frac{iu_{0\omega}}{2}\int_0^\infty e^{-2kz\xi}(1 + \xi^2)\xi \;\times$$

$$\times \left\{ \frac{\mu(\zeta^* - \zeta)}{|\mu\xi + \zeta|^2} + \frac{\varepsilon\zeta^* - \varepsilon^*\zeta}{|\varepsilon\xi + \zeta|^2} \right\} d\xi\;. \tag{I.18}$$

Let us estimate $u_{\omega\,\text{quas}}$ at very large and very small values of z.

If $kz \gg \left| \frac{\varepsilon}{\sqrt{\varepsilon\mu - 1}} \right|$, then in the expression under the integral sign we can everywhere limit ourselves to the lowest degree in ξ (so that $\zeta = i\sqrt{\varepsilon\mu - 1}$ and the denominators are equal to $|\zeta|^2$). We then obtain

$$u_{\omega\,\mathrm{quas}} \approx \frac{u_0\omega}{2}\left(\frac{\mu+\varepsilon}{\sqrt{\varepsilon\mu-1}} + \frac{\mu+\varepsilon^*}{\sqrt{\varepsilon^*\mu-1}}\right)\int_0^\infty e^{-2kz\xi}\,\xi\,d\xi =$$

$$= \frac{u_0\omega}{8(kz)^2}\left(\frac{\mu+\varepsilon}{\sqrt{\varepsilon\mu-1}} + \frac{\mu+\varepsilon^*}{\sqrt{\varepsilon^*\mu-1}}\right). \qquad \text{(I.19)}$$

For small kz, we shall roughly approximate the expression under the integral sign by the substitution

$$\zeta = \sqrt{\xi^2 + 1 - \varepsilon\mu} \to \xi + \sqrt{1-\varepsilon\mu} = \xi + \zeta_0 \qquad \text{(I.20)}$$

Then (I.18) becomes the following expression

$$u_{\omega\,\mathrm{quas}} = \frac{iu_0\omega}{2}\int_0^\infty e^{-2kz\xi}(\xi^3+\xi) \times$$

$$\times \left\{ \frac{\mu(\zeta_0^* - \zeta_0)}{|(\mu+1)\xi + \zeta_0|^2} + \frac{(\varepsilon - \varepsilon^*)\xi + \varepsilon\zeta_0^* - \varepsilon^*\zeta_0}{|(\varepsilon+1)\xi + \zeta_0|^2} \right\}.$$

Separating out the positive powers of ξ and carrying out the integration, we obtain the main terms, since the remaining integrals of the form

$$\int_0^\infty e^{-2kz\xi}\,\frac{(A\xi + B)\,d\xi}{a\xi^2 + b\xi + c}$$

give only a logarithmic dependence on kz. The main terms are these:

$$u_{\omega\,quas} \approx \frac{iu_0\omega}{8}\left\{\frac{\varepsilon - \varepsilon^*}{|\varepsilon + 1|^2(kz)^3} + \right.$$

$$\left. + \left[\frac{\varepsilon^*\zeta_0^*}{(\varepsilon^* + 1)^2} - \frac{\varepsilon\zeta_0}{(\varepsilon + 1)^2} + \frac{\mu(\zeta_0^* - \zeta_0)}{(\mu + 1)^2}\right]\frac{1}{(kz)^2} + O\left(\frac{1}{(kz)}\right)\right\} . \quad (I.21)$$

Thus, near the surface itself $u_{\omega\,quas}$ grows as $1/z^3$.

If the radiating medium has a very high conductivity ($|\varepsilon| \gg 1$), then introducing the thickness of the skin-layer $d = c/\sqrt{2\pi\sigma\mu\nu}$, which is related to ε in the following manner:

$$\sqrt{\frac{\varepsilon}{\mu}} = \frac{1 - i}{k\mu d} = \frac{1 - i}{\delta}, \quad (\delta \ll 1) \quad (I.22)$$

we obtain from (I.19)

$$u_{\omega\,quas} = \frac{u_0\omega}{4k^2z^2\delta}, \quad (kz \gg 1/\delta), \quad (I.23)$$

and from (I.21)

$$u_{\omega\,quas} = \frac{u_0\omega}{4}\left\{\frac{\delta^2}{\mu(kz)^3} + \frac{2\mu^2}{(\mu + 1)^2\delta(kz)^2} + O\left(\frac{1}{(kz)}\right)\right\} .$$

The first term has a very small coefficient and is predominant only for $kz \ll \delta^3$. For $kz \gg \delta^3$, the following term is predominant, which if we change $\mu + 1$ to μ[1] becomes identical with expression (I.23) for very large distances.

Let us note, that for very large $|\varepsilon|$, when it can be considered that

$$\zeta = \sqrt{\xi^2 + 1 - \varepsilon\mu} \approx i\sqrt{\varepsilon\mu} = \frac{(1 + i)\mu}{\delta} .$$

The general formula (I.18) becomes the following:

1. $\mu + 1$ is obtained because of the accepted rough approximation (I.20).

$$u_{quas} = \frac{u_0}{2}\int_0^{\infty} e^{-2kz\xi} \times$$

$$\times\ (1+\xi^2)\xi\left\{\frac{1}{\xi^2 - \xi + \frac{2}{2}} + \frac{1}{\frac{2\xi^2}{2} + \xi + 1}\right\} d\xi \quad (I.24)$$

and from here we obtain an estimate of (I.23) both for $kz \gg 1/$ and $kz \ll$.

For complex values of ε_1, i.e., in the case of an absorbing external medium, the normal component of the wave vector (I.16) is complex for any real values of p_1 and p_2 and the separation of the field in the external medium into travelling and uniform standing waves is lost. Formula (I.16) for the density of energy flow remains valid, but now the separation of a finite domain for integration does not take place. This means that in the absorbing medium waves with any p_1 and p_2 participate in the energy flow formed -- both those which become travelling waves as $\varepsilon_1'' \to 0$ as well as those which become non-uniform. As p_1^2 and p_2^2 increase, the exponential multiplier in (I.10) takes the form: $\exp(-2\sqrt{p_1^2 + p_2^2}\, z)$ and guarantees the convergence of the integral (finite energy flow) for any $z > 0$. In order to obtain the corresponding expressions for the intensity I_ω, (I.10) must be transformed to the form (5.7), i.e., to a solid angle integral.

For a single plane wave in the half-space $z > 0$, we have

$$e^{i(p_1x+p_2y+\varepsilon_3 z)} = e^{i(p_1x+p_2y+\frac{\varepsilon_3+\varepsilon_3^*}{2}z)}\, e^{i\frac{\varepsilon_3-\varepsilon_3^*}{2}z}$$

wherefrom it is clear that the absolute value of a wave vector will now be a function of the angle θ between its direction and the z axis. Denoting this absolute value of the wave vector by $P = P(\theta)$, we have in lieu of (I.12),

$$p_1 = P\sin\theta\cos\varphi, \quad p_2 = P\sin\theta\sin\varphi,$$
$$\frac{t_3 + t_3^*}{2} = -P\cos\theta, \tag{I.25}$$

wherefrom

$$p_1^2 + p_2^2 = P^2\sin^2\theta,$$

$$dp_1\,dp_2 = \frac{\partial(p_1, p_2)}{\partial(\theta,\varphi)}\,d\theta\,d\varphi = \frac{1}{2}\frac{\partial(P^2\sin^2\theta)}{\partial\theta}\,d\theta\,d\varphi\,. \tag{I.26}$$

Denoting, for brevity,

$$k^2\varepsilon_1\mu_1 = k^2\mu_1(\varepsilon_1' - i\varepsilon_1'') = q_1^2, \quad k^2\varepsilon\mu = q^2 \tag{I.27}$$

so that

$$t_3 = -\sqrt{q_1^2 - (p_1^2 + p_2^2)} = -\sqrt{q_1^2 - P^2\sin^2\theta}\,,$$
$$s_3 = \sqrt{q^2 - P^2\sin^2\theta}\,. \tag{I.28}$$

From the last equality in (I.25) we obtain for P the equation

$$P^4 - \frac{q_1^2 + q_1^{*2}}{2}P^2 + \frac{(q_1^2 - q_1^{*2})^2}{16\cos^2\theta} = 0$$

whose solution, becoming for real q_1 $P = q_1$, is

$$P = \frac{1}{2}\sqrt{q_1^2 + q_1^{*2} + \sqrt{(q_1^2 + q_1^{*2})^2 - \frac{(q_1^2 - q_1^{*2})^2}{\cos^2\theta}}}\ .$$

$$= k\sqrt{\frac{\mu_1}{2}\left(\sqrt{\varepsilon_1'^2 + \frac{\varepsilon_1''^2}{\cos^2\theta}} + \varepsilon_1'\right)}. \qquad (I.29)$$

Upon varying θ from 0 to $\pi/2$, the wave number P, determined by the real part of the wave phase, varies from the value

$$kn = k\sqrt{\frac{\mu_1}{2}\left(\sqrt{\varepsilon_1'^2 + \varepsilon_1''^2} + \varepsilon_1'\right)}$$

to ∞. From (I.25) and (I.29) it is not difficult to see that variation of p_1 and p_2 from $-\infty$ to $+\infty$ corresponds to varying θ and φ within the limits of the positive hemisphere, i.e., $0 \leq \varphi \leq 2\pi$, $0 \leq \theta \leq \pi/2$.

Formula (I.10) (to whose denominator must be added μ_1), upon carrying out the substitution (I.26) in it and substituting into it the value of C, takes the form

$$S_{\omega 3} = -\frac{\mu^2(\varepsilon^* - \varepsilon)I_{0\omega}}{4\mu_1}\int_0^{2\pi} d\varphi \int_0^{\pi/2} d\theta \,\frac{\partial P^2 \sin^2\theta}{\partial\theta}\,\times$$

$$\times\ \frac{t_3+t_3^*}{s_3^*-s_3}\, e^{i(t_3-t_3^*)z}\left(\frac{1}{|\alpha|^2} + \frac{(P^2\sin^2\theta + |t_3|^2)(P^2\sin^2\theta + |s_3|^2)}{|\Delta|^2}\right).$$

Introducing the solid angle element $d\Omega = \sin\theta\, d\theta\, d\varphi$ and utilizing the fact that

$$t_3 + t_3^* = -P\cos\theta, \qquad s_3^* - s_3 = \frac{k^2\mu(\varepsilon^* - \varepsilon)}{s_3^* + s_3}$$

we further obtain

$$s_{\omega 3} = \frac{\mu I_{0\omega}}{\mu_1 k^2} \int_{\theta < \pi/2} \cos\theta \, d\Omega \times$$

$$\times \left(\frac{\partial P}{\partial \theta} \sin\theta + P \cos\theta \right) P^2 (s_3^* + s_3) \, e^{i(t_3 - t_3^*)z} \times$$

$$\times \left(\frac{1}{|\alpha|^2} + \frac{(P^2 \sin^2\theta + |t_3|^2)(P^2 \sin^2\theta + |s_3|^2)}{|\Delta|^2} \right)$$

wherefrom, in accordance with (5.7), we obtain

$$I_{\omega} = \frac{\mu I_{0\omega}}{\mu_1 k^2} \left(\frac{\partial P}{\partial \theta} \sin\theta + P \cos\theta \right) P^2 \, (s_3^* + s_3) \, e^{i(t_3 - t_3^*)z} \times$$

$$\times \left(\frac{1}{|\alpha|^2} + \frac{(P^2 \sin^2\theta + |t_3|^2)(P^2 \sin^2\theta + |s_3|^2)}{|\Delta|^2} \right). \qquad (I.30)$$

Here

$$\alpha = \frac{\mu}{\mu_1} t_3 - s_3, \qquad \Delta = t_3 s^2 - \frac{\mu}{\mu_1} s_3 t^2 .$$

Since $t^2 = q_1^2$ and $s^2 = q^2$, the expression for Δ is brought to the following form with the aid of (I.28)

$$\Delta = k^2 \varepsilon \mu \left(t_3 - \frac{\varepsilon_1}{\varepsilon} s_3 \right) .$$

According to (I.30), the intensity is exponentially attenuated with z. For the index of extinction $i(t_3^* - t_3)$, using (I.27) - (I.29), we find

$$i(t_3 - t_3^*) = -\sqrt{2\mu_1\left(\sqrt{\varepsilon_1'^2 + \frac{\varepsilon_1''^2}{\cos^2\theta}} - \varepsilon_1'\right)} .$$

Hence it can be seen that attenuation is smallest in the direction of the normal to the surface ($\theta = 0$). In this direction the intensity is equal to

$$I_\omega = I_{0\omega}\frac{2(\sqrt{\varepsilon\mu} + \sqrt{\varepsilon^*\mu})}{|\sqrt{\varepsilon_1\mu} + \sqrt{\varepsilon\mu_1}|^2}\left[\frac{\mu_1}{2}\left(\sqrt{\varepsilon_1'^2 + \varepsilon_1''^2} + \varepsilon_1'\right)\right]^{3/2} \times$$

$$\times \exp\left\{ -z\sqrt{2\mu_1(\sqrt{\varepsilon_1'^2 + \varepsilon_1''^2} - \varepsilon_1')}\right\} .$$

II. Derivation of Formulae (7.6) and (7.14)

Let us introduce the notation

$$e^{2it_3\ell} = \sigma . \tag{II.1}$$

From (7.4) and (7.5) we then have

$$w_1 = -\sigma v_1, \quad w_2 = -\sigma v_2, \quad w_3 = \sigma v_3 \tag{II.2}$$

which permits us immediately to eliminate $\vec{w}$. Equation (7.3) then assumes the form

$$u_1 - v_1(1 - \sigma) = \frac{\varepsilon - 1}{\varepsilon}\int_{-\infty}^{+\infty}\frac{dp_3}{p_3^2 - s_3^2} \times$$

$$\times \{(p_1^2 - \sigma^2)s_1 + p_1 p_2 s_2\} ,$$

$$s_2 - v_2(1 - \sigma) = \frac{\varepsilon - 1}{\varepsilon} \int_{-\infty}^{+\infty} \frac{dp_3}{p_3^2 - s_3^2} \times$$

$$\times \; [p_1 \, p_2 \, g_1 + (p_2^2 - s^2) g_2],$$

$$p_1 u_3 - s_3 u_1 - \mu(1 + \sigma)(p_1 v_3 - t_3 v_1) =$$

$$= - \frac{\varepsilon - 1}{\varepsilon} p_1 s^2 \int_{-\infty}^{+\infty} \frac{g_3 dp_3}{p_3^2 - s_3^2}$$

$$p_2 u_3 - s_3 u_2 - \mu(1 + \sigma)(p_2 v_3 - t_3 v_2) =$$

$$= - \frac{\varepsilon - 1}{\varepsilon} p_2 s^2 \int_{-\infty}^{+\infty} \frac{g_3 dp_3}{p_3^2 - s_3^2} .$$

Adding here the equations (7.5) for u and v

$$p_1 u_1 + p_2 u_2 + s_3 u_3 = 0, \qquad p_1 v_1 + p_2 v_2 + t_3 v_3 = 0$$

and eliminating u, we obtain for v exactly the same equations as in (I.2). The distinction consists of the fact that in lieu of (I.3) and (I.4) we now have

$$\alpha = \mu t_3(1 + \sigma) - s_3(1 - \sigma)$$

$$\beta = t_3(1 - \sigma) - \mu s_3(1 + \sigma) \tag{II.3}$$

$$\Delta = (p_1^2 + p_2^2)\beta - s_3 t_3 \alpha = t_3(1 - \sigma)s^2 - \mu s_3(1 + \sigma)t^2 .$$

Formulas (I.5) - (I.11) are valid in the same way, but with the meanings

for α, β and Δ given by (II.3).

Going over to polar coordinates in (I.11) we now have

$$\alpha = -k[\mu(1+\sigma)\cos\theta + (1-\sigma)\sqrt{\varepsilon\mu - \sin^2\theta}\,]$$

$$\Delta = -k^3\mu[\varepsilon(1-\sigma)\cos\theta + (1+\sigma)\sqrt{\varepsilon\mu - \sin^2\theta}\,].$$

Let us note that all formulae in section 6 and Appendix I are formally obtainable from those examined here if we let $\sigma = 0$. This is understandable since σ is proportional to the reflection coefficient of the mirror (taken to be unity in our case). Taking into account that

$$1+\sigma = 2e^{it_3\ell}\cos t_3\ell, \qquad 1-\sigma = -2ie^{it_3\ell}\sin t_3\ell,$$

we can further write

$$\alpha = -2ke^{-i\xi}(\mu\cos\theta\cos\xi + i\sqrt{\varepsilon\mu - \sin^2\theta}\,\sin\xi)$$

$$\Delta = -2k^3\mu e^{-i\xi}(i\varepsilon\cos\theta\sin\xi + \sqrt{\varepsilon\mu - \sin^2\theta}\,\cos\xi)$$

where

$$\xi = -t_3\ell = k\ell\cos\theta. \tag{II.4}$$

If we use the value for C given by (6.15), these expressions transform (I.11) to the form (7.6).

Using (7.2), (II.1) and II.2), we obtain from (7.1) the following expression for the components H_1 and H_2 on the surface of the mirror for $z = \ell$:

$$\begin{aligned} H_1 &= -\frac{2}{k}\int_{-\infty}^{+\infty}(p_2v_3 - t_3v_2)e^{i(p_1x+p_2y+t_3\ell)}\,dp_1\,dp_2 \\ H_2 &= \frac{2}{k}\int_{-\infty}^{+\infty}(p_1v_3 - t_3v_1)e^{i(p_1x+p_2y+t_3\ell)}\,dp_1\,dp_2 \end{aligned} \tag{II.5}$$

With the help of expression (I.5) for the components of v, we find

$$p_1 v_3 - t_3 v_1 = -\frac{k^2 \mu (\varepsilon - 1) s_3}{\alpha \Delta} \int_{-}^{+} \frac{dp_3}{p_3^2 - s_3^2} \{ [\alpha s_3 (p_1^2 + t_3^2) - \beta p_2^2 t_3] g_1 +$$

$$+ p_1 p_2 (\alpha s_3 + \beta t_3) g_2 + \gamma p_2 t^2 g_3 \} , \tag{II.6}$$

$$p_2 v_3 - t_3 v_2 = -\frac{k^2 \mu (\varepsilon - 1) s_3}{\alpha \Delta} \int_{-}^{+} \frac{dp_3}{p_3^2 - s_3^2} \{ p_1 p_2 (\alpha s_3 + \beta t_3) g_1 +$$

$$+ [\alpha s_3 (p_2^2 + t_3^2) - \beta p_1^2 t_3] g_3 + \gamma p_2 t^2 g_3 \} .$$

Since the correlation function of the components of g (5.12) contains $\delta(\vec{p} - \vec{p}\,')$, we can identify p and p' in all multipliers containing $\delta(\vec{p} - \vec{p}\,')$. Therefore, according to (II.5)

$$\overline{|H_1|^2} + \overline{|H_2|^2} = \frac{4}{k^2} \iint_{-}^{+} \overline{|p_1 v_3 - t_3 v_1|^2} +$$

$$+ \overline{|p_2 v_3 - t_3 v_2|^2} \; e^{i(t_3 - t_3^*)} \, dp_1 \, dp_2 \, dp_1' \, dp_2' . \tag{II.7}$$

Further, (II.6) and (6.12) give

$$\overline{|p_1 v_3 - t_3 v_1|^2} + \overline{|p_2 v_3 - t_3 v_2|^2} =$$

$$= \frac{k^4 \mu^2 |\varepsilon - 1|^2 |s_3|^2 2C}{(2\pi)^3 |\alpha|^2 |\Delta|^2} \delta(p_1 - p_1') \, \delta(p_2 - p_2') A \iint_{-}^{+} \frac{\delta(p_3 - p_3') \, dp_3 \, dp_3'}{|p_3^2 - s_3^2|^2} \tag{II.8}$$

where the following notation is introduced

$$A = |\alpha s_3 (p_1^2 + t_3^2) - \beta p_2^2 t_3|^2 + |\alpha s_3 (p_2^2 + t_3^2) - \beta p_1^2 t_3|^2 +$$

$$+ 2p_1^2\, p_2^2 |\alpha s_3 + \beta t_3|^2 + |\alpha|^2 |t|^4 (p_1^2 + p_2^2) .$$

Developing this expression and using formulae (6.6), (7.2) and (II.3), we can show that

$$A = |t_3|^2 |\Delta|^2 + |t|^4 |\alpha|^2 |s|^2 . \qquad \text{(II.9)}$$

The integral in (II.8) then immediately reduces to a single integral (I.9). In sum, after substituting (I.9) and (II.9) into (II.8) we obtain

$$\overline{|p_1 v_3 - t_3 v_1|^2} + \overline{|p_2 v_3 - t_3 v_2|^2} =$$

$$= \frac{ik^4 \mu^2 |\varepsilon - 1|^2 C}{(2\pi)^2 (s_3^* - s_3)} \left\{ \frac{|t_3|^2}{|\alpha|^2} + \frac{|s|^2 |t|^4}{|\Delta|^2} \right\} \delta(p_1 - p_1')\,\delta(p_2 - p_2').$$

Carrying this into (II.7), and the result into (7.13), we find

$$S = \frac{ik^2 \mu^2 |\varepsilon - 1|^2 C}{4\pi^3} \sqrt{\frac{\omega}{2\pi\sigma}} \int_{-\infty}^{+\infty} \frac{e^{i(t_3 - t_3^*)\ell}}{s_3^* - s_3} \cdot \left\{ \frac{|t_3|^2}{|\alpha|^2} + \frac{|s|^2 |t|^4}{|\Delta|^2} \right\} dp_1\, dp_2 .$$

Multiply and divide the expression under the integral sign by $s_3^* + s_3$. Taking into account that

$$s_3^{*2} - s_3^2 = k^2 \mu (\varepsilon^* - \varepsilon)$$

substituting the value of C from (6.15) and going over to polar coordinates ($p_1 = p \cos\varphi$, $p_2 = p \sin\varphi$, $p_1^2 + p_2^2 = p^2$, $dp_1\, dp_2 = 2\pi p\, dp$), we obtain

$$S = \frac{4\pi\mu I_0 \omega}{k^3} \sqrt{\frac{\omega}{2\pi\sigma}} \int_0^{\infty} e^{i(t_3 + t_3^*)\ell} (s_3^* + s_3) \left\{ \frac{|t_3|^2}{|\alpha|^2} + \frac{k^4 (p^2 + |s_3|^2)}{|\Delta|^2} \right\} p\, dp \qquad \text{(II.10)}$$

where α and Δ are expressed by formulae (II.3) and t_3 and s_3, according to (6.6) and (6.8) equal

$$t_3 = -\sqrt{k^2 - p^2}, \quad s_3 = \sqrt{k^2 \varepsilon\mu - p^2}.$$

The integration interval in (II.10) can be split into two: from 0 to k, in which t_3 is real (travelling waves) and from k to ∞, in which t_3 is purely imaginary (non-uniform standing waves). Letting $p = k \sin\theta$ in the first interval and $p = k \cosh\varphi$ in the second interval and introducing the notation

$$\left.\begin{aligned} a &= \sqrt{\varepsilon\mu - \sin^2\theta}, \\ \xi &= k\ell\cos\theta, \end{aligned}\right\} \quad \left.\begin{aligned} b &= \sqrt{\varepsilon\mu - \cosh^2\varphi}, \\ \eta &= k\ell\sinh\varphi, \end{aligned}\right\}$$

we transform (II.10) to the form (7.14).

III. Derivation of formulae (8.11), (8.15), (8.24) and (8.28)

Substituting expansions (8.2) and (8.3) into expressions (8.8) and (8.10) for S_ω, $u_{e\omega}$ and $u_{m\omega}$, we obtain

$$S_\omega = -\frac{c}{4\pi k\mu}\left\{\iint_{-\infty}^{+\infty} \overline{[\vec{a}\,[\vec{p}', \vec{a}^*]]}\, e^{i(\vec{p}-\vec{p}', \vec{r})}\, d\vec{p}\, d\vec{p}' + \text{conjugate}\right\},$$

$$u_{e\omega} = \frac{\varepsilon + \varepsilon^*}{8\pi}\iint_{-\infty}^{+\infty} \overline{\vec{a}\vec{a}^*}\, e^{i(\vec{p}-\vec{p}', \vec{r})}\, d\vec{p}\, d\vec{p}',$$

$$u_{m\omega} = \frac{1}{4\pi k^2\mu}\iint_{-\infty}^{+\infty} \overline{([\vec{p}, \vec{a}][\vec{p}', \vec{a}^*])}\, e^{i(\vec{p}-\vec{p}', \vec{r})}\, d\vec{p}\, d\vec{p}'.$$

Taking into account that the correlation function of the components of $\vec{a}$, i.e., (8.7) contains the multiplier $\delta(\vec{p} - \vec{p}')$, we can let $\vec{p} = \vec{p}'$ in all expressions under the integral sign. Further

$$[\vec{a}[\vec{p}, \vec{a}^*]] = \vec{p}(\vec{a}, \vec{a}^*) - \vec{a}^*(\vec{p}, \vec{a}),$$

$$[\vec{p}, \vec{a}][\vec{p}, \vec{a}^*] = p^2(\vec{a}, \vec{a}^*) - (\vec{p}, \vec{a})(\vec{p}, \vec{a}^*)$$

so that

$$s_\omega = -\frac{c}{4\pi k\mu} \iint_{-\infty}^{+\infty} \left\{ 2\vec{p}\overline{(\vec{a}, \vec{a}^*)} - \overline{\vec{a}(\vec{p}, \vec{a}^*)} - \overline{\vec{a}^*(\vec{p}, \vec{a})} \right\} d\vec{p}\, d\vec{p}' .$$

$$u_{e\omega} = \frac{\varepsilon + \varepsilon^*}{8\pi} \iint_{-\infty}^{+\infty} \overline{\vec{a}\,\vec{a}^*}\, d\vec{p}\, d\vec{p}' \tag{III.1}$$

$$u_{m\omega} = \frac{1}{4\pi k^2\mu} \iint_{-\infty}^{+\infty} \left\{ p^2\, \overline{(\vec{a}, \vec{a}^*)} - \overline{(\vec{p}, \vec{a})(\vec{p}, \vec{a}^*)} \right. d\vec{p}\, d\vec{p}' .$$

Using now (8.6) and (8.7), it is convenient to apply this method of notation of double summation with respect to the two indices encountered, for example

$$\overline{(\vec{p}, \vec{a})(\vec{p}, \vec{a}^*)} = p_\alpha p_\beta \overline{a_\alpha a_\beta^*} =$$

$$= \frac{|\varepsilon - 1|^2 \delta(\vec{p} - \vec{p}')}{(2\pi)^3 |\varepsilon|^2} p_\alpha p_\beta \left\{ U(p) \frac{k^4 |\varepsilon|^2 \mu^2 \delta_{\alpha\beta} - \vec{p}_\alpha \vec{p}_\beta [k^2(\varepsilon + \varepsilon^*)\mu - p^2]}{|p^2 - k^2\varepsilon\mu|^2} - \right.$$

$$\left. - V(p) \frac{p_\alpha p_\beta}{p^2} \right\} = \frac{|\varepsilon - 1|^2 \delta(\vec{p} - \vec{p}')}{(2\pi)^3 |\varepsilon|^2} p^2 \left\{ U(p) - V(p) \right\}$$

we obtain

$$s_\omega = -\frac{2k^3 \varepsilon\mu |\varepsilon - 1|^2}{(2\pi)^4} \int_{-\infty}^{+\infty} \frac{U(p)\, \vec{p}\, d\vec{p}}{|p^2 - k^2\varepsilon\mu|^2}$$

$$u_{e\omega} = \frac{k^4(\varepsilon + \varepsilon^*)\mu^2 |\varepsilon - 1|^2}{2(2\pi)^?} \int_{-\infty}^{+\infty} \frac{U(p)\, d\vec{p}}{|p^2 - k^2\varepsilon\mu|^2} +$$

$$+ \frac{1}{2k^4|\epsilon|^2\mu^2}\int_{-\infty}^{+\infty}[U(p) - V(p)\, d\vec{p}\Big\},$$

$$u_{m\omega} = \frac{k^2\mu|\epsilon - 1|^2}{(2\pi)^4}\int_{-\infty}^{+\infty}\frac{U(p)\,p^2\,d\vec{p}}{|p^2 - k^2\epsilon\mu|^2}.$$

Formula (8.11) follows from here as a result of passing in the p-space to polar coordinates ($d\vec{p} = p^2 dp d\Omega$), whereby the expression for the intensity I_ω is obtained as a result of comparing $S_{\omega N}[\vec{N}\vec{p} = p\cos\vartheta]$ with (5.7).

With $V(p) = U(p) = C$, only the integral $\int_0^\infty \frac{p^2 dp}{|p^2 - k^2\epsilon\mu|^2}$ remains in expression (8.11) for $u_{e\omega}$ which can readily be evaluated with the help of deductions (see below). The condition $V(p) = U(p)$, in view of (4.9), means that

$$\nabla^2\Psi + \Phi = 0 \tag{III.2}$$

and, consequently, for $\Phi(r) = C\delta(r)$ $\Psi(r)$ has the form of Coulomb's potential

$$\Psi = \frac{C}{r}.$$

For the correlation function of the components of $\vec{K}$

$$F_{\alpha\beta} = \Phi(r)\delta_{\alpha\beta} + \frac{\partial^2\Psi(r)}{\partial x_\alpha \partial x_\beta}$$

condition (III.2) leads to the following

$$\frac{\partial F_{\alpha\beta}}{\partial x_\beta} = \frac{\partial}{\partial x_\alpha}(\Phi + \nabla^2\Psi) = 0.$$

Since

$$\frac{\partial F_{\alpha\beta}}{\partial x_\beta} = -\frac{\partial}{\partial x''_\beta}\overline{E_\alpha(\vec{r}')\,E^*_\beta(\vec{r}'')} = -E_\alpha(\vec{r}')\frac{\partial E^*_\beta(\vec{r}'')}{\partial x''_\beta},$$

the rendering of the divergence of the tensor $F_{\alpha\beta}$ to zero necessarily entrains

$$\frac{\partial E_\beta}{\partial x_\beta} = \operatorname{div}\vec{E} = 0. \qquad \text{(III.3)}$$

But, according to (1.7) and (1.8), we have

$$\rho = \frac{1}{4\pi}\operatorname{div}\vec{E} = -\frac{\varepsilon-1}{4\pi\varepsilon}\operatorname{div}\vec{E}$$

and, thus, (III.3) denotes the absence of charges.

Substituting expression (8.14) into (8.11)

$$U(p) = Ce^{-\kappa^2 p^2/4}, \quad V(p) = 0$$

and using the value of C

$$C = \frac{8\pi^3(\varepsilon^* - \varepsilon)}{k^3ci|\varepsilon - 1|^2}I_{0\omega} = \frac{2\pi^2(\varepsilon^* - \varepsilon)}{k^3i|\varepsilon - 1|^2}u_{0\omega},$$

we obtain

$$I_\omega = A\int_0^\infty \frac{e^{-\rho^2x^2}x^3dx}{X(x)}, \quad u_{e\omega} = B_1\left\{\int_0^\infty \frac{e^{-\rho^2x^2}x^2dx}{X(x)} + \frac{1}{2}\int_0^\infty e^{-\rho^2x^2}x^2dx\right\},$$

$$u_{m\omega} = B_2\int_0^\infty \frac{e^{-\rho^2x^2}x^4dx}{X(x)} = B_2\left\{\int_0^\infty e^{-\rho^2x^2}dx + \right. \qquad \text{(III.4)}$$

$$\left. + 2\cos 2\eta\int_0^\infty \frac{e^{-\rho^2x^2}x^2dx}{X(x)} - \int_0^\infty \frac{e^{-\rho^2x^2}dx}{X(x)}\right\},$$

where the following notation has been introduced

$$q = k\sqrt{\varepsilon\mu} = kn(1 - i\varkappa) = |q|e^{-i\eta}, \quad \rho = \frac{|q|a}{2},$$

$$x = \frac{p}{|q|}, \quad X(x) = x^4 - 2x^2 \cos 2\eta + 1, \tag{III.5}$$

$$A = I_{0\omega}\frac{kn^2\varkappa}{\pi}, \quad B_1 = u_{0\omega}\frac{2n^3\varkappa(1 - \varkappa^2)}{\pi\sqrt{1 + \varkappa^2}}, \quad B_2 = u_{0\omega}\frac{2n^3\varkappa\sqrt{1 + \varkappa^2}}{\pi}. \tag{III.6}$$

Two zeros $X(x)$ lie in the upper half plane

$$x = e^{i\eta} \quad \text{and} \quad x = -e^{-i\eta}.$$

Making deductions in these poles, we find that the integrals

$$\int_0^{\infty} \frac{dx}{X(x)} = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dx}{X(x)} \quad \text{and} \quad \int_0^{\infty} \frac{x^2dx}{X(x)} = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{x^2dx}{X(x)}$$

have the same value, namely

$$\frac{\pi}{4\sin\eta} = \frac{\pi\sqrt{1 + \varkappa^2}}{4\varkappa}.$$

Therefore,

$$\int_0^{\infty} \frac{e^{-\rho^2x^2}dx}{X(x)} = \int_0^{\infty} \frac{dx}{X(x)} + O(\rho^2) = \frac{\pi\sqrt{1 + \varkappa^2}}{4\varkappa} + O(a^2) \tag{III.7}$$

$$\int_0^{\infty} \frac{e^{-\rho^2x^2}x^2dx}{X(x)} = \int_0^{\infty} \frac{x^2dx}{X(x)} + O(\rho) = \frac{\pi\sqrt{1 + x^2}}{4\varkappa} + O(a). \tag{III.8}$$

Taking further into account that

$$\int_0^{\infty} e^{-\rho^2x^2}dx = \frac{\sqrt{\pi}}{2\rho}, \quad \int_0^{\infty} e^{-\rho^2x^2}x^2dx = \frac{\sqrt{\pi}}{4\rho^3},$$

and expressing with the help of (III.5) and (III.6) ρ, η, B_1 and B_2

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in terms of n and $\varkappa$, we obtain formula (8.15) for $u_{e\omega}$ and $u_{\varkappa\omega}$.

The evaluation of the intensity I_ω is complicated by the unevenness of the integrand which makes it impossible to spread the integral over the entire axis. Performing the change of integration variable $x^2 = z$, we can write

$$I_\omega = \frac{A}{2} \lim_{\varepsilon \to 0} \int_\varepsilon^\infty \frac{e^{-\rho^2 z}\, z\, dz}{X}.$$

Adding and subtracting under the lim sign $\int_\varepsilon^\infty \frac{e^{-\rho^2 z}\, dz}{z}$, we obtain

$$I_\omega = \frac{A}{2} \lim_{\varepsilon \to 0} \left\{ 2\cos 2\eta \int_\varepsilon^\infty \frac{e^{-\rho^2 z}\, dz}{X} - \int_\varepsilon^\infty \frac{e^{-\rho^2 z}\, dz}{Xz} + \int_\varepsilon^\infty \frac{e^{-\rho^2 z}\, dz}{z} \right\} . \tag{III.9}$$

In the first integral we can immediately let $\varepsilon \to 0$, so that it reduces to (III.7). The second integral converges for $\rho = 0$, so that

$$\int_\varepsilon^\infty \frac{e^{-\rho^2 z}\, dz}{Xz} = \int_\varepsilon^\infty \frac{dz}{Xz} + O(\rho^2) = -\frac{1}{2} \ln \frac{\varepsilon^2}{1 - 2\varepsilon \cos 2\eta + \varepsilon^2} +$$

$$+ \frac{1}{\operatorname{tg} 2\eta} \left(\frac{\pi}{2} + \operatorname{arctg} \frac{\cos 2\eta - \varepsilon}{\sin 2\eta} \right) + O(\varepsilon^2) .$$

The third integral is

$$\int_\varepsilon^\infty \frac{e^{-\rho^2 z}\, dz}{z} = -\operatorname{Ei}(\rho^2 \varepsilon) = -\ln \gamma \rho^2 + O(\rho^4 \varepsilon^2).$$

Substitution into (III.9), passage to the limit $\varepsilon \to 0$ and replacement with the help of (III.5) and (III.6) of A, ρ and η by their expressions in terms of n and $\varkappa$ give formula (8.15) for I_ω .

For brevity in writing, let us introduce the notation

$$u = \frac{q^2}{r} - \frac{iq}{r^2} - \frac{1}{r^3}, \qquad v = \frac{iq}{r^2} + \frac{1}{r^3}. \tag{III.10}$$

Expression (8.23) for $\vec{E}$ and $\vec{H}$ can then be written in the form

$$\vec{E} = \frac{\varepsilon - 1}{4\pi\varepsilon} \int \left\{ \vec{K}u - \vec{r}(\vec{r}, \vec{K}) \frac{u - 2v}{r^2} \right\} e^{-iqr}\, dV,$$

$$\vec{H} = -\frac{ik(\varepsilon - 1)}{4\pi} \int [\vec{r}, \vec{K}]\, v e^{-iqr}\, dV. \tag{III.11}$$

With the help of the correlation function (8.17) it is not difficult to obtain the following equation

$$\overline{\vec{K}(\vec{r})\vec{K}^*(\vec{r}\,')} = 3Cf(R), \qquad \overline{\vec{K}^*(\vec{r}\,')(\vec{r}\,', \vec{K}(\vec{r}))} = \vec{r}\,'Cf(R),$$

$$\overline{(\vec{r}\,', \vec{K}(\vec{r}))(\vec{r}, \vec{K}^*(\vec{r}\,'))} = (\vec{r}, \vec{r}\,')\, Cf(R), \quad \text{etc.} \tag{III.12}$$

Substituting now (III.11) into formula (8.8) for the density of energy flow, we obtain

$$S_\omega = \frac{ikc|\varepsilon - 1|^2}{64\pi^3\varepsilon} \iint \left\{ uv^* [\vec{r}\,' \overline{(\vec{K}, \vec{K}^*)} - \overline{\vec{K}^* (\vec{r}\,', \vec{K})}] - \right.$$

$$- \frac{(u - 2v)v^*}{r^2} [\vec{r}\,' \overline{(\vec{r}, \vec{K})(\vec{r}, \vec{K}^*)} -$$

$$\left. - (\vec{r}, \vec{r}\,') \overline{\vec{K}^* (\vec{r}, \vec{K})}] \right\} e^{i(q^* r' - qr)}\, dV\, dV' + \text{conjugate}$$

where u, v, $\vec{K}$ refer to the point $\vec{r}$, and u^*, v^*, $\vec{K}^*$ to the point $\vec{r}\,'$.

Utilizing formulae (III.12), we further find

$$S_\omega = Aiq^* \int e^{-iqr} \left\{ (u + 2v)\vec{P} + \frac{\vec{r}(u - 2v)}{r^2} (\vec{r}, \vec{P}) \right\} dV + \text{conjugate} \tag{III.13}$$

where

$$A = \frac{kc|\varepsilon - 1|^2 c}{64\pi^3|\varepsilon|^2}, \quad \vec{F} = \int f(R) e^{iq^* r'} v^* \vec{r}' \, dV' . \qquad (III.14)$$

In the case of the δ-correlation, when

$$f(R) = \delta(\vec{r} - \vec{r}')$$

we have from (III.14)

$$\vec{F} = e^{iq^* r} v^* \vec{r}$$

and (III.13) gives

$$S_\omega = 2Ai \int (\varepsilon^* uv^* - \varepsilon u^* v) e^{i(q^* - q)r} \vec{r} \, dV .$$

For the energy flow through a unit surface, whose normal $\vec{N}$ forms the angle θ with $\vec{r}$, if we take into account that

$$dV = r^2 \, dr d\Omega, \quad \vec{r}\vec{N} = r \cos\theta \qquad (III.15)$$

we obtain

$$S_{\omega N} = \int_{\theta < \pi/2} \cos\theta \, d\Omega \cdot 2Ai \int_0^\infty e^{i(q^* - q)r} (\varepsilon^* uv^* - \varepsilon u^* v) r^3 \, dr .$$

Comparison with (5.7) shows that the intensity is

$$I_\omega = 2Ai \int_0^\infty e^{i(q^* - q)r} (\varepsilon^* uv^* - \varepsilon u^* v) r^3 \, dr .$$

Substituting here the expression (III.10) for u and v, the value of the constant A from (III.14) and performing simple transformations, we obtain the first of formulae (8.24).

Analogous evaluations, carried out by substituting (III.11) into formula (8.10) for $u_{e\omega}$ and $u_{m\omega}$, and using equation (III.12) in the result, gives

$$u_{e\omega} = \frac{(\varepsilon^* + \varepsilon)|\varepsilon - 1|^2 c}{64\pi^3|\varepsilon|^2} \int e^{i(q^*-q)r} (uu^* + 2vv^*)dV$$

$$u_{m\omega} = \frac{k^2\mu|\varepsilon - 1|^2 c}{32\pi^3} \int e^{i(q^*-q)r} vv^* r^2 dV .$$

Letting here $dV = 4\pi r^2 dr$ and introducing u and v from (III.10), we obtain the remaining two formulae of (8.24).

In the case of the correlation function (8.27) let us take as coordinate axis of the spherical system the direction $\vec{r}$. The direction of the normal $\vec{N}$ to the unit surface is determined in this coordinate system by the angles θ and φ, and the direction $\vec{r}'$ by the angles ξ and η (Fig. 24). Beside (III.15), we have

$$\vec{r}\vec{r}' = \vec{r}\vec{r}' \cos\xi, \qquad dV' = \vec{r}'^2 dr' \sin\xi d\xi d\eta$$
$$\vec{N}\vec{r}' = \vec{r}' [\sin\theta \sin\xi \cos(\eta - \varphi) + \cos\theta \cos\xi] \tag{III.16}$$

and from (III.13) and (III.14) we obtain for the component $S_{\omega N}$

$$S_{\omega N} = \int_{\theta < \pi/2} d\Omega \cdot Ai\varepsilon^* \int_0^\infty e^{-iqr} r^2 dr \Big\{ 2u \cos\theta \times$$
$$\times \int f(R) e^{iq^*r'} v^* r'^3 dr' \int \cos\xi \sin\xi d\xi \int_0^{2\pi} d\eta + \tag{III.17}$$
$$+ (u + 2v) \sin\theta \int f(R) e^{iq^*r'} v^* r'^3 dr' \int \sin^2\xi d\xi \int_0^{2\pi} \cos(\eta - \varphi) d\eta \Big\} +$$
$$+ \text{conjugate.}$$

Upon integration with respect to η the second term disappears, and comparison of the first term with (5.7) gives the following formula for the intensity

$$I_{\omega} = 4\pi A i \varkappa^{*} \int_{0}^{\infty} e^{-iqr} u r^{2} F(r)\, dr + \text{conjugate} \qquad (III.18)$$

where

$$F(r) = \int f(R)\, e^{iq^{*}r'} v^{*} r'^{3}\, dr' \int \cos\xi \sin\xi\, d\xi .$$

The limits of integration for ξ and r' are here established by the condition that $f(R) \neq 0$ only for

$$R^{2} = r^{2} + r'^{2} - 2rr' \cos\xi < a^{2} .$$

As can easily be seen from Fig. 24, this condition gives the following integration limits

$$\text{For } r > a: \quad \text{for } r' \text{ from } r - a \text{ to } r + a, \quad \text{for } \xi \text{ from } 0 \text{ to } \xi_m$$

$$\text{For } r < a: \begin{cases} \text{for } r' \text{ from } a - r \text{ to } a + r, & \text{for } \xi \text{ from } 0 \text{ to } \xi_m \\ \text{for } r' \text{ from } 0 \text{ to } a - r, & \text{for } \xi \text{ from } 0 \text{ to } \pi \end{cases}$$

$$(III.19)$$

where

$$\cos\xi_m = \frac{r^{2} + r'^{2} - a^{2}}{2rr'} .$$

Since $\int_{0}^{\pi} \sin\xi \cos\xi\, d\xi = 0$, only the first two lines of (III.19) remain and we obtain

$$F(r) = -\frac{3}{32\pi a^{3} \varkappa^{2}} \int_{|r-a|}^{r+a} e^{iq^{*}r'} v^{*} r'^{3} \left[\frac{(r^{2} - a^{2})^{2}}{r'^{2}} - 2(r^{2} + a^{2}) + r'^{2} \right] dr' .$$

Substituting v from (III.10) and integrating we further get

$$
F(r) = -\frac{3}{2\pi a^3 r^2}\begin{cases} e^{iq^*a}\,(1 - iq^*a)\,\dfrac{\partial}{\partial q^*}\left(\dfrac{\sinh iq^*a}{q^*}\right) & (r < a), \\[2ex] e^{iq^*r}\,(1 - iq^*r)\,\dfrac{\partial}{\partial q^*}\left(\dfrac{\sinh iq^*r}{q^*}\right) & (r > a). \end{cases} \tag{III.20}
$$

Denoting $iq^* = \alpha\ (-iq = \alpha^*)$, substituting into (III.18) the expression (III.10) for u and (III.20) for $F(r)$, we get

$$
I_\omega = \frac{6Ai\varepsilon^*}{a^3\alpha}\left\{e^{\alpha a}(1-\alpha a)\int_0^a e^{\alpha^* r}\left(|\alpha|^2 r^2 + \alpha^* r - 1\right)\frac{\partial}{\partial\alpha}\left(\frac{\sinh\alpha r}{\alpha}\right)\frac{dr}{r^3} + \right.
$$

$$
\left. + \frac{\partial}{\partial\alpha}\left(\frac{\sinh\alpha a}{\alpha}\right)\int_a^\infty e^{(\alpha+\alpha^*)r}\left(|\alpha|^2 r^2 + \alpha^* r - 1\right)(1-\alpha r)\frac{dr}{r^3}\right\} +
$$

$$
+ \text{conjugate} .
$$

The first term can be directly expanded in positive powers of a. In the second term integration by parts permits separation of negative powers of a from the integral exponent which for small a gives a logarithmic term and a positive power series in a. Using then expression (III.14) for A, substituting the expression of C in terms of $I_{0\omega}$ and going from ε and $\alpha = iq^* = ik\sqrt{\varepsilon^*}$ (we assume $\mu = 1$) to n and $\varkappa$ by means of (8.13), we obtain formula (8.28).

IV. Derivation of formulae (10.16), (10.17), (10.19) and (10.22) - (10.24)

To find A, B, ..., $\widetilde{C}$ as a function of r, let us apply the method of variation of constants to (10.6). Taking (10.5) into account, substitute (10.6) and (10.8) into equation (10.1). This gives

$$
[\nabla A, \vec{N}] + [\nabla B, \vec{M}] + [\nabla\widetilde{A}, \vec{\widetilde{N}}] + [\nabla\widetilde{B}, \vec{\widetilde{M}}] =
$$

$$
= q\left\{C\vec{L} + \widetilde{C}\vec{\widetilde{L}} + \vec{G}e^{i(n\varphi + hz)}\right\} , \tag{IV.1}
$$

$$[\nabla A, \vec{N}] + [\nabla B, \vec{N}] + [\nabla \widetilde{A}, \widetilde{\vec{N}}] + [\nabla \widetilde{B}, \widetilde{\vec{N}}] +$$
$$+ [\nabla \widetilde{C}, \widetilde{\vec{L}}] + [\nabla \widetilde{C}, \widetilde{\vec{L}}] = 0 .$$

But the gradients A, B, ..., $\widetilde{C}$ have a component only with respect to $\vec{i}_1$ ($\nabla A = \vec{i}_1 A'$ etc., where, as before, the prime indicates differentiation with respect to r). Therefore equations (IV.1), if we rewrite them with the help of (20.3) into components, become

$$CJ' + \widetilde{C}N' = -G_r$$

$$\lambda^2(A'J + \widetilde{A}'N) + \frac{inq^2}{r}(CJ + \widetilde{C}N) = -q^2 G_\varphi ,$$

$$\frac{hn}{qr}(A'J + \widetilde{A}'N) + (B'J' + \widetilde{B}'N') +$$
$$+ ihq(CJ + \widetilde{C}N) = -qG_z , \qquad \text{(IV.2)}$$

$$\lambda^2(B'J + \widetilde{B}'N) + ihq(C'J + \widetilde{C}'N) = 0 ,$$

$$(A'J' + \widetilde{A}'N') + \frac{hn}{qr}(B'J + \widetilde{B}'N) -$$
$$- \frac{in}{r}(C'J + \widetilde{C}'N) = 0 .$$

Here $J = J_{|n|}(\lambda r)$, $N = N_{|n|}(\lambda r)$ are Bessel and Neumann functions.

For six quantities we have five equations. A supplementary condition can be imposed. Let us take the following relation for such a condition

$$CJ + \widetilde{C}N = 0 \qquad \text{(IV.3)}$$

by virtue of which C and $\widetilde{C}$ drop out from $E_{0\varphi}$ and E_{0z}. Differentiating (IV.3), we obtain in view of the first equation of (IV.2)

$$C'J + \widetilde{C}'N = G_r . \qquad \text{(IV.4)}$$

The remaining four equations of (IV.2) take the form

$$A'J + \widetilde{A}'N = -\frac{q^2}{\lambda^2} G_\varphi ,$$

$$\frac{hn}{qr}(A'J + \widetilde{A}'N) + (B'J' + \widetilde{B}'N') = -qG_z ,$$

$$B'J + \widetilde{B}'N = -\frac{ihq}{\lambda^2} G_r ,$$

$$(A'J' + \widetilde{A}'N') + \frac{hn}{qr}(B'J + \widetilde{B}'N) = \frac{in}{r} G_r .$$

From these equations, using the relation

$$JN' - J'N = \frac{2}{\pi r} , \tag{IV.5}$$

we find

$$\begin{aligned} A' &= -\frac{\pi q^2 r}{2\lambda^2}\left(G_\varphi N' + \frac{in}{r} G_r N\right) , \\ B' &= -\frac{\pi q r}{2}\left[\frac{ih}{\lambda^2} G_r N' + \left(G_z - \frac{hn}{\lambda^2 r} G_\varphi\right) N\right] , \\ \widetilde{A}' &= \frac{\pi q^2 r}{2\lambda^2}\left(G_\varphi J' + \frac{in}{r} G_r J\right) , \\ \widetilde{B}' &= \frac{\pi q r}{2}\left[\frac{ih}{\lambda^2} G_r J - \left(G_r - \frac{hn}{\lambda^2 r} G_\varphi\right) J\right] . \end{aligned} \tag{IV.6}$$

With the first equation of (IV.2), the condition (IV.3) and equation (IV.6) the question of the dependence of A, B, ..., $\widetilde{C}$ on r is solved.

Further, we must express the constants P and Q of the internal field in terms of the coefficients A, B, ... of the primary field. For this substitute (10.6), (10.10) and (10.11) into the boundary conditions (10.13). This gives the following four equations:

$$AJ' + \widetilde{A}N' + \frac{hn}{qa}(BJ + \widetilde{B}N) + A_1J' +$$

$$+ \frac{hn}{qa}B_1J = PH' + \frac{hn}{ka}QH,$$

$$\frac{\lambda^2}{q}(BJ + \widetilde{B}N + B_1J) = \frac{\lambda_0^2}{k}QH\ ,$$

$$\gamma\left[BJ' + \widetilde{B}N' + \frac{hn}{qa}(AJ + \widetilde{A}N) + B_1J' +\right. \tag{IV.7}$$

$$\left. + \frac{hn}{qa}A_1J\right] = i\left(QH' + \frac{hn}{ka}PH\right),$$

$$\frac{\gamma\lambda^2}{q}(AJ + \widetilde{A}N + A_1J) = \frac{i\lambda_0^2}{k}PH\ ,$$

where here and further on we shall understand by J and N functions with the argument λa, by H functions with the argument $\lambda_0 a$ and a prime indicates differentiation with respect to a. Eliminating the constants A_1 and B_1 from (IV.7), we obtain for P and Q the equations

$$\Delta_1 P + \delta Q = \frac{2\widetilde{A}}{\pi a}$$

$$\delta P + \Delta_2 Q = \frac{2\gamma\widetilde{B}}{i\pi a} \tag{IV.8}$$

where

$$\Delta_1 = H'J - \frac{iq\lambda_0^2}{\gamma k\lambda^2}HJ', \quad \Delta_2 = H'J - \frac{\gamma q\lambda_0^2}{ik\lambda^2}HJ'\ ,$$

$$\delta = \frac{hn}{ka}\left(1 - \frac{\lambda_0^2}{\lambda^2}\right)HJ\ . \tag{IV.9}$$

Solving equation (IV.8), we find

$$P = \frac{2}{\pi a \Delta}(\Delta_2 \tilde{A} + i\gamma\delta\tilde{B}), \qquad Q = -\frac{2}{\pi a \Delta}(\delta\tilde{A} + i\gamma\Delta_1\tilde{B}), \tag{IV.10}$$

where

$$\Delta = \Delta_1 \Delta_2 - \delta^2 . \tag{IV.11}$$

P and Q contain only $\tilde{A}$ and $\tilde{B}$ (values for $r = a$), which in the developed form are expressed according to (IV.6) in the following manner:

$$\tilde{A} = \frac{\pi q^2}{2\lambda^2}\int_0^a [G_{n\varphi}(r, h)J'_{|n|}(\lambda r) + \frac{in}{r} G_{nr}(r, h)J_{|n|}(\lambda r)]\, r dr ,$$

$$\tilde{B} = \frac{\pi q}{2}\int_0^a \left\{ \frac{ih}{\lambda^2} G_{nr}(r, h)\, J'_{|n|}(\lambda r) - [G_{nz}(r, h) - \frac{hn}{\lambda^2 r} G_{n\varphi}(r, h)]\, J_{|n|}(\lambda r)\right\} r dr . \tag{IV.12}$$

Let us evaluate the power, emitted by unit length of cylinder

$$P_\omega = r\int_{-\pi}^{+\pi} S_{\omega r}\, d\varphi = \frac{cr}{4\pi}\int_{-\pi}^{+\pi}\left\{ E_\varphi H_z^* - E_z H_\varphi^* + E_\varphi^* H_z - E_z^* H_\varphi \right\} d\varphi .$$

Substituting in here (10.11), one should consider that

$$\int_{-\pi}^{+\pi} e^{i(n-m)\varphi}\, d\varphi = 2\pi\delta_{nm} ,$$

that $\lambda_0^2 = k^2 - h^2$ is a real quantity and that $\overline{|P|^2}$ and $\overline{|Q|^2}$ contain, as we shall see, the multiplier $\delta(h - h_1)$, in view of which we can let $h = h_1$ in all expressions under the sign of integration.

$$P_\omega = \frac{icr}{2k}\sum_{n=-\infty}^{+\infty}\iint_{-\infty}^{+\infty} dh dh_1 \lambda_0^2 (H'H^* - H'^*H)(\overline{|P|^2} + \overline{|Q|^2}) . \tag{IV.13}$$

It is not difficult to convince oneself that the multiplier $H'H^* - H'^*H$ reduces the region of integration with respect to h to the interval $(-k, +k)$. In fact[1]

$$H'H^* - H'^*H = \lambda_0 H_{n-1}^{(2)} (\lambda_0 r) H_n^{(1)} (\lambda_0^* r) - \lambda_0^* H_{n-1}^{(1)} (\lambda_0^* r) H_n^{(2)} (\lambda_0 r).$$

But λ_0 is either real ($|h| \leqslant k$) or imaginary ($|h| > k$). In the former case $H_n^{(1)} (\lambda_0 r) = J_n + iN_n$, $H_n^{(2)} (\lambda_0 r) = J_n - iN_n$ and we obtain

$$H'H^* - H'^*H = 2i\lambda_0 (J_{n-1} N_n - J_n N_{n-1}) = -\frac{4i}{\pi r}, \qquad (|h| \leqslant k). \qquad (IV.14)$$

In the second case $\lambda_0^* r = -\lambda_0 r$, so that[2] $H_n^{(2)} (\lambda_0 r) = J_n - iN_n$, $H_n^{(1)} (-\lambda_0 r) = (-1)^{n+1} (J_n - iN_n)$, and therefore

$$H'H^* - H'^*H = 0, \qquad (|h| > k). \qquad (IV.15)$$

By virtue of (IV.14) and (IV.15), expression (IV.13) for emitted power takes the form (10.16)

$$P_\omega = \frac{2c}{\pi k} \sum_{n=-\infty}^{+\infty} \int_{-k}^{+k} dh \int_{-\infty}^{+\infty} dh_1 \lambda_0^2 (\overline{|P|^2} + \overline{|Q|^2}) . \qquad (IV.16)$$

Let us note that the restricting of the integration interval in h to the limits $(-k, +k)$ is analogous to the corresponding narrowing down of the integration region in the plane case studied above and, as before, denotes diffraction "smoothing" of the field structure on the surface, i.e., limitation of P_ω only by those components of the Fourier integral in z on the cylinder surface which are determined by the travelling waves.

1. Further on we shall omit the absolute magnitude sign in the index of cylinder functions, i.e., instead of $|n|$, we shall simply write n, remembering however that the index is taken as positive.

2. G. N. Watson, "Theory of Bessel Functions", p. 90 (M. 1949).

Substituting expansion (IV.10) into (IV.16) for P and Q, we find

$$P_\omega = \frac{8c}{k\gamma^3 a^2}\sum_{n=-\infty}^{+\infty}\int_{-k}^{+k} dh \int_{-\infty}^{+\infty} dh_1 \frac{\lambda_0^2}{|\Delta|^2}\Big\{ \overline{|\tilde{A}|^2}\,(|\Delta_2|^2 + |\delta|^2) +$$

$$+ |\gamma|^2\overline{|\tilde{B}|^2}\,(|\Delta_1|^2 + |\delta|^2) + \overline{\tilde{A}^*\tilde{B}}\, i\gamma(\Delta_2^*\delta + \Delta_1\delta^*) -$$

$$- \overline{\tilde{A}\tilde{B}^*}\, i\gamma^*(\Delta_2\delta^* + \Delta_1^*\delta)\Big\} . \qquad \text{(IV.17)}$$

But from (IV.12), applying the correlation function (10.15), we obtain

$$\overline{|\tilde{A}|^2} = \frac{|q|^2|\varepsilon - 1|^2 C}{16|\varepsilon|^2|\lambda|^4} \cdot \frac{|q|^2(\lambda^2 JJ'^* - \lambda^{*2}J^*J')\alpha}{\lambda^2 - \lambda^{*2}}\,\delta(h - h_1) , \qquad \text{(IV.18)}$$

$$\overline{|\tilde{B}|^2} = \frac{|q|^2|\varepsilon - 1|^2 C}{16|\varepsilon|\cdot|\lambda|^4} \cdot \frac{h^2(\lambda^2 JJ'^* - \lambda^{*2}J^*J') + |\lambda|^4(JJ'^* - J^*J')}{\lambda^2 - \lambda^{*2}}\,\alpha\delta(h - h_1)$$

$$\overline{\tilde{A}\tilde{B}^*} = \frac{|q|^2|\varepsilon - 1|^2 C}{16|\varepsilon|^2|\lambda|^4} \cdot anh\, JJ^*\delta(h - h_1) .$$

Carrying these expressions into (IV.17), noting that $|q|^2 = k^2|\varepsilon|\mu$, $|\gamma|^2 = \frac{|\varepsilon|}{\mu}$, $i\gamma q^* = -k|\varepsilon| = -i\gamma^* q$, and substituting the value (9.5) for the constant C, we arrive at formula (10.17)

$$P_\omega = \frac{4\mu(\varepsilon^* - \varepsilon)\, I_{0\omega}}{ik^2a^2}\sum_{n=-\infty}^{+\infty}\int_{-k}^{+k}\frac{\lambda_0^2\, dh}{|\lambda|^4|\Delta|^2}\; \times$$

$$\times \Big\{\frac{a}{\lambda^2 - \lambda^{*2}}\Big[k^2\mu(|\Delta_2|^2 + |\delta|^2)(\lambda^2 JJ'^* - \lambda^{*2}J^*J') +$$

$$+ \frac{1}{\mu}(|\Delta_1|^2 + |\delta|^2)(h^2(\lambda^2 JJ'^* - \lambda^{*2}J^*J') + |\lambda|^4(JJ'^* - J^*J'))\Big] -$$

$$- ikhn JJ^*\,[(\Delta_2^* + \Delta_1^*)\delta + (\Delta_2 + \Delta_1)\delta^*]\Big\} . \qquad \text{(IV.19)}$$

Let us find the coefficients of absorption of the cylinder for P and Q waves with fixed values of n and h. For the field internal to the cylinder, we have in accordance with (10.10)

$$\vec{E}_1 = A_1\vec{M} + B_1\vec{N}, \qquad \vec{H}_1 = \gamma(A_1\vec{N} + B_1\vec{M}). \qquad (IV.20)$$

The reflected field is expressed, according to (10.11), in the following way

$$\vec{E} = P\vec{M}^* + Q\vec{N}^*, \qquad \vec{H} = i(P\vec{N}^* + Q\vec{M}^*) \qquad (IV.21)$$

and the field of incident waves differs from (IV.21) by other constants (P_0, Q_0) and by the fact that $H \equiv H_n^{(2)}(\lambda_0 r)$ is replaced by $H^* \equiv H_n^{(1)}(\lambda_0 r)$ (we immediately assume that only travelling waves are taken, i.e., $|h| < k$ and, therefore, λ_0 is real). Thus, the field strengths of incident waves are

$$\vec{E}_0 = P_0\vec{M}^* + Q_0\vec{N}^*, \qquad \vec{H}_0 = i(P_0\vec{N}^* + Q_0\vec{M}^*) \qquad (IV.22)$$

where the curl denotes functions of $\vec{M}^*$ and $\vec{N}^*$ in which H is replaced by H^*.

The boundary conditions (1.13) now have the form

$$\begin{aligned} E_{1\varphi} &= E_{0\varphi} + E_\varphi, & H_{1\varphi} &= H_{0\varphi} + H_\varphi, \\ E_{1z} &= E_{0z} + E_z, & H_{1z} &= H_{0z} + H_z \end{aligned} \qquad (r = a)$$

and after substituting into them (IV.20) - (IV.22), they give the following equations:

$$A_1' J + \frac{hn}{qa} B_1 J = P_0 H'^* + \frac{hn}{ka} Q_0 H^* + PH' + \frac{hn}{ka} QH,$$

$$\frac{\lambda^2}{q} B_1 J = \frac{\lambda_0^2}{k}(Q_0 H^* + QH),$$

$$\gamma\left(B_1 J' + \frac{hn}{qa} A_1 J\right) = i\left(Q_0 H'^* + \frac{hn}{ka} P_0 H^* + QH' + \frac{hn}{ka} PH\right),$$

$$\frac{\gamma\lambda^2}{q} A_1 J = -\frac{i\lambda_0^2}{k}\left(P_0 H^* + PH\right).$$

Eliminating A_1 and B_1 with the help of the second and fourth equations, we obtain from the first and third equations

$$\begin{aligned} \Delta_1 P + \delta Q &= -(\varphi_1 P_0 + \psi Q_0), \\ \delta P + \Delta_2 Q &= -(\psi P_0 + \varphi_2 Q_0), \end{aligned} \qquad \text{(IV.23)}$$

where

$$\varphi_1 = H'^* J - \frac{iq\lambda_0^2}{\gamma k \lambda^2} H^* J', \qquad \varphi_2 = H'^* J - \frac{\gamma q \lambda_0^2}{ik\lambda^2} H^* J',$$
$$\psi = \frac{hn}{ka}\left(1 - \frac{\lambda_0^2}{\lambda^2}\right) H^* J, \qquad \text{(IV.24)}$$

and Δ_1, Δ_2 and δ are given by formulae (IV.9).

Solving (IV.23) for P and Q, we find

$$\begin{aligned} P &= \frac{1}{\Delta}\left[(\psi\delta - \varphi_1\Delta_2)P_0 + (\varphi_2\delta - \psi\Delta_2)Q_0\right], \\ Q &= \frac{1}{\Delta}\left[(\varphi_1\delta - \psi\Delta_1)P_0 + (\psi\delta - \varphi_2\Delta_1)Q_0\right]. \end{aligned} \qquad \text{(IV.25)}$$

From here it can be seen that the incidence of only one of the P_0- (or Q_0) waves gives in the reflected field both P- and Q waves. At high conductivities (for a symmetrical wave $n = 0$ always) P- and Q-waves do not get mixed up upon reflection. We are interested in the absorption coefficients in the presence of an incident wave, either P_0 or Q_0, i.e.,

$$A_{Pn}(h) = 1 - \frac{(|P|^2 + |Q|^2)_{Q_0=0}}{|P_0|^2}$$

$$A_{Qn}(h) = 1 - \frac{(|P|^2 + |Q|^2)_{P_0=0}}{|Q_0|^2}$$

Let us carry out the evaluation for $A_{Pn}(h)$ only since the evaluation for $A_{Qn}(h)$ is completely analogous.

Substituting expression (IV.25) into $A_{Pn}(h)$, we obtain

$$A_{Pn}(h) = \frac{1}{|\Delta|^2}\left\{|\Delta_1\Delta_2 - \delta^2|^2 - |\psi\delta - \varphi_1\Delta_2|^2 - |\varphi_1\delta - \psi\Delta_1|^2\right\} .$$

Adding and subtracting within the brackets $|\delta|^2|\Delta_1|^2$ and taking into account that $|\psi|^2 = |\delta|^2$, we transform $A_{Pn}(h)$ to the form

$$A_{Pn}(h) = \frac{1}{|\Delta|^2}\Big\{(|\Delta_2|^2 + |\delta|^2)(|\Delta_1|^2 - |\varphi_1|^2) +$$

$$+ (\varphi_1\psi^* - \delta^*\Delta_1)(\delta\Delta_1^* + \delta^*\Delta_2) + (\varphi_1^*\psi - \delta\Delta_1^*)(\delta^*\Delta_1 + \delta\Delta_2^*)\Big\} .$$

But with the help of (IV.9), (IV.14) and (IV.24) it is not difficult to obtain the following

$$\varphi_1\psi^* - \delta^*\Delta_1 = \frac{4ihn}{\pi ka^2}\left(1 - \frac{\lambda_0^2}{\lambda^2}\right) JJ^* ,$$

$$|\Delta_1|^2 - |\varphi_1|^2 = \frac{4i\lambda_0^2}{\pi a|\lambda|^4}(\lambda^2 JJ'^* - \lambda^{*2}J^*J') .$$

Finally

$$A_{Pn}(h) = \frac{4i\lambda_0^2}{\pi a^2|\Delta|^2}\Big\{\frac{a\mu}{|\lambda|^4}(|\Delta_2|^2 + |\delta|^2)(\lambda^2 JJ'^* - \lambda^{*2}J^*J') +$$

$$+ \frac{hn}{k} JJ^* \left[\left(\frac{1}{\lambda_0^2} - \frac{1}{\lambda^{*2}}\right)(\delta\Delta_1^* - \delta^*\Delta_2)\left(\frac{1}{\lambda_0^2} - \frac{1}{\lambda^2}\right)(\delta^*\Delta_1 - \delta\Delta_2^*)\right]\Big\} .$$

Analogously, we find

$$A_{Qn}(h) = \frac{4i\lambda_0^2}{\pi a^2|\Delta|^2}\left\{ \frac{a(|\Delta_1|^2 + |\delta|^2)}{|\lambda|^4 k^2\mu} [h^2(\lambda^2 JJ'^* - \lambda^{*2}J^*J') + \right.$$

$$+ |\lambda|^4(JJ'^* - J^*J')] + \frac{hn}{k} JJ^* \left[\left(\frac{1}{\lambda_0^2} - \frac{1}{\lambda^{*2}}\right)(\delta^*\Delta_1 + \delta\Delta_2^*) - \right.$$

$$\left.\left. - \left(\frac{1}{\lambda_0^2} - \frac{1}{\lambda^2}\right)(\delta\Delta_1^* - \delta^*\Delta_2)\right]\right\} .$$

Joining this to the preceding expression, we obtain

$$A_{Pn}(h) + A_{Qn}(h) = \frac{4i\lambda_0^2(\lambda^2 - \lambda^{*2})}{\pi k^2 a^2|\Delta|^2|\lambda|^4}\left\{ \dots \right\} , \qquad (IV.26)$$

where inside the flower brackets we obtain the same thing as in (IV.19). Substitution of these flower brackets from the last expression into (IV.19), keeping in mind that $\lambda^2 - \lambda^{*2} = q^2 - q^{*2} = k^2\mu(\varepsilon - \varepsilon^*)$, gives formula (10.19)

$$P_\omega = \frac{2\pi^2 I_{0\omega}}{k^2} \sum_{n=-\infty}^{+\infty} \int_{-k}^{+k} [A_{Pn}(h) + A_{Qn}(h)] \frac{dh}{2\pi} . \qquad (IV.27)$$

Let us evaluate the emitted power for several special cases. Let us examine the case of a well-conducting cylinder, i.e., highly developed skin-effect

$$a \gg d = c/\sqrt{2\pi\mu\sigma\omega} \qquad (IV.28)$$

(d is the thickness of the skin-layer). In this case the refraction index

$$\sqrt{\varepsilon\mu} \approx \sqrt{-i\varepsilon''\mu} = \frac{1-i}{kd} \qquad (IV.29)$$

is very large in absolute value, and since $|h|$ does not exceed k, we can consider that $\lambda^2 = k^2\varepsilon\mu - h^2 \approx k^2\varepsilon\mu$. Thereby, $|\lambda|a = \sqrt{2}\,\frac{a}{d} \gg 1$, and we can use the asymptotic expressions for the Bessel functions. [1]

$$J = J_n(\lambda a) = \sqrt{\frac{2}{\pi\lambda a}}\cos\left[\lambda a - \frac{(2n+1)\pi}{4}\right]$$

$$J' = \frac{dJ_n(\lambda a)}{da} = -\sqrt{\frac{2\lambda}{\pi a}}\sin\left[\lambda a - \frac{(2n+1)\pi}{4}\right].$$

Taking into account also that

$$\operatorname{tg}\left[\lambda a - \frac{(2n+1)\pi}{4}\right] \approx -i$$

we obtain

$$\lambda^2 JJ'^* - \lambda^{*2}J^*J' = \frac{2|\lambda|(\lambda+\lambda^*)}{i\pi a}\left|\cos\left[\lambda a - \frac{(2n+1)\pi}{4}\right]\right|^2$$

$$JJ'^* - J^*J' = \frac{2(\lambda+\lambda^*)}{i\pi a|\lambda|}\left|\cos\left[\lambda a - \frac{(2n+1)\pi}{4}\right]\right|^2$$

Further, according to (IV.9),

$$\Delta_1 = \sqrt{\frac{2}{\pi\lambda a}}\,H\left(\frac{H'}{H} + \frac{\mu\lambda_0^2}{i\lambda}\right)\cos\left[\lambda a - \frac{(2n+1)\pi}{4}\right]$$

$$\Delta_2 = \sqrt{\frac{2}{\pi\lambda a}}\,H\left(\frac{H'}{H} + \frac{\varepsilon\lambda_0^2}{i\lambda}\right)\cos\left[\lambda a - \frac{(2n+1)\pi}{4}\right]$$

$$\delta = \sqrt{\frac{2}{\pi\lambda a}}\,H\,\frac{hn}{kd}\cos\left[\lambda a - \frac{(2n+1)\pi}{4}\right].$$

Substituting all these expressions into $A_{Pn}(h) + A_{Qn}(h)$ [see (IV.26) and (IV.19)], we get

1. G. N. Watson, p. 218.

$$A_{Pn}(h) + A_{Qn}(h) = \frac{4i\lambda_0^2(\varepsilon - \varepsilon^*)}{\pi k^2 a|\varepsilon||H|^2} \frac{1}{\left|\rho\sigma - \left(\frac{hn}{ka}\right)^2\right|^2} \times$$

$$\times \left\{ \frac{k\sqrt{\mu}(\sqrt{\varepsilon} + \sqrt{\varepsilon^*})}{i(\varepsilon - \varepsilon^*)} \left[|\sigma|^2 + \frac{|\varepsilon|}{\mu}\left(|\rho|^2 + \left(\frac{hn}{ka}\right)^2\right)\right] - \right.$$

$$\left. - \left(\frac{hn}{ka}\right)^2 \frac{\rho + \rho^* + \sigma + \sigma^*}{|\varepsilon|\mu} \right\} \qquad \text{(IV.30)}$$

where the following notations have been introduced

$$\rho = \frac{H'}{H} + \frac{\mu\lambda_0^2}{i\lambda}, \qquad \sigma = \frac{H'}{H} + \frac{\varepsilon\lambda_0^2}{i\lambda}. \qquad \text{(IV.31)}$$

<u>Thin, well conducting cylinder</u>

$$ka \ll 1 \qquad \text{(IV.32)}$$

but as before $|\lambda|a \gg 1$, i.e., $a \gg d$. In this case, since also $\lambda_0 a \ll 1$, we can utilize the following expressions[1]

$$H_n(\lambda_0 a) = \frac{i(n-1)!}{\pi}\left(\frac{2}{\lambda_0 a}\right)^n, \qquad H_n'(\lambda_0 a) = -\frac{in!}{\pi a}\left(\frac{2}{\lambda_0 a}\right)^n \quad (n \neq 0)$$

$$\text{(IV.33)}$$

$$H_0(\lambda_0 a) = -\frac{2i}{\pi}\ln\frac{\lambda_0 a}{2}, \qquad H_0'(\lambda_0 a) = -\frac{2i}{\pi a}.$$

It is not difficult to convince oneself that $A_{Pn}(h) + A_{Qn}(h) \sim a^{2|n|}$ and thus we can limit ourselves to the zero term of the series (IV.27), which corresponds to a symmetrical wave ($n = 0$). From (IV.30) we obtain

$$A_{P_0}(h) + A_{Q_0}(h) = \frac{4\lambda_0^2\sqrt{\mu}(\sqrt{\varepsilon} + \sqrt{\varepsilon^*})}{\pi ka|\varepsilon||H_0|^2}\left(\frac{1}{|\rho|^2} + \frac{|\varepsilon|}{\mu|\sigma|^2}\right).$$

1. G. N. Watson, pp 51 and 75.

With the help of (IV.29), (IV.31) and (IV.33) this expression becomes

$$A_{P_0}(h) + A_{Q_0}(h) = \pi \lambda_0^2 a \mu d \left\{ \left| 1 - \frac{i\lambda_0^2 a \mu d}{1 - i} \ln \frac{\lambda_0 a}{2} \right|^{-2} + \left| \frac{k \mu a}{1 - i} - \frac{i\lambda_0^2 a}{k} \ln \frac{\lambda_0 a}{2} \right|^{-2} \right\} .$$

Substituting this into (IV.27), introducing in lieu of h the integration variable θ

$$h = k \sin \theta, \qquad \lambda_0 = \sqrt{k^2 - h^2} = k \cos \theta ,$$

we get

$$P_{\omega} = 2\pi^2 k a \mu d I_{0\omega} \int_0^{\pi/2} \left\{ \left| 1 - \frac{ik^2 a \mu d}{1 - i} \cos^2\theta \ln \left(\frac{ka \cos \theta}{2} \right) \right|^{-2} + \left| \frac{k \mu d}{1 - i} - ika \cos^2\theta \ln \left(\frac{ka \cos \theta}{2} \right) \right|^{-2} \right\} \cos^3\theta \, d\theta .$$

For an estimate, we can neglect $\cos \theta$ under the log sign and in the first term of the expression under the integration sign we can neglect the second part of the denominator. Then

$$P_{\omega} = 2\pi^2 k a \mu d I_{0\omega} \int_0^{\pi/2} \left\{ 1 + \frac{1}{\frac{k^2 \mu^2 d^2}{2} - k\mu d \alpha \cos^2\theta + \alpha^2 \cos^4\theta} \right\} \cos^3\theta \, d\theta ,$$

where $\alpha = ka \ln \frac{ka}{2}$. The integral of the first term equals 2/3, of the second (taking into account that $|\alpha| \gg k\mu d$) approximately to

$\frac{\pi \sin \pi/8}{\sqrt[4]{2}\sqrt{k\mu d |\alpha|^3}} \approx \frac{1}{\sqrt{k\mu d|\alpha|^3}}$. In sum, neglecting 2/3, we get

$$P_\omega = \frac{2\pi^2 I_{0\omega}}{k}\sqrt{\frac{\mu d}{a\left|\ln\frac{ka}{2}\right|^3}} . \tag{IV.34}$$

This power is obtained from unit cylinder length, i.e., from the surface $2\pi a$. From unit cylinder surface for $I_{0\omega} = \frac{\theta k^2}{4\pi^3}$ (Rayleigh-Jeans law) the power

$$P_{S\omega} = \frac{P_\omega}{2\pi a} = \frac{\theta k}{4\pi^2 a}\sqrt{\frac{\mu d}{a\left|\ln\frac{ka}{2}\right|^3}} \tag{IV.35}$$

is emitted.

Thick, well conducting cylinder

$ka \gg 1$.

In this case for $|n| \ll ka$ we can utilize the approximate expressions[1]

$$H_n(\lambda_0 a) \approx \sqrt{\frac{2}{\pi\lambda_0 a}}\, e^{-i\left[\lambda_0 a - \frac{(2n+1)\pi}{4}\right]}$$

$$H_n'(\lambda_0 a) \approx -i\sqrt{\frac{2\lambda_0}{\pi a}}\, e^{-i\left[\lambda_0 a - \frac{(2n+1)\pi}{4}\right]} . \tag{IV.36}$$

For $|n| \gg ka$ expression (IV.30) decreases very rapidly with increasing n. Therefore we shall assume for a rough estimate that summation with respect to n in (IV.27) extends to values of n from $-\chi ka$ to $+\chi ka$, where χ is a coefficient of the order of 1.

From (IV.31) and (IV.36) we get

$$\rho = -i\lambda_0 + \frac{\mu\varepsilon\lambda_0^2}{i\lambda} \approx -i\lambda_0, \quad \sigma = -i\lambda_0 + \frac{\varepsilon\lambda_0^2}{i\lambda} \approx \frac{\varepsilon\lambda_0^2}{i\lambda} .$$

1. G. N. Watson, p. 220.

Substituting this into (IV.30) and continuing to retain only the senior with respect to $\sqrt{\varepsilon}$ terms, we find

$$A_{Pn}(h) + A_{Qn}(h) = \frac{2}{k}\left(\sqrt{\frac{\varepsilon}{\mu}} + \sqrt{\frac{\varepsilon^*}{\mu}}\right) \frac{\lambda_0^3\left[\frac{\lambda_0^4}{k} + \lambda_0^2 + \left(\frac{hn}{ka}\right)^2\right]}{\left|\frac{\varepsilon\lambda_0^3}{\lambda} + \left(\frac{hn}{ka}\right)^2\right|^2} \qquad \text{(IV.37)}$$

Let us now introduce $k = k \sin\theta (\lambda_0 = k\cos\theta)$ and $y = n/ka$, and let us go from summation with respect to n to integration with respect to y

$$\sum_{n=-\chi ka}^{+\chi ka} \longrightarrow ka \int_{-\chi}^{+\chi} dy.$$

Substitution of (IV.37) into (IV.27) then gives

$$P_\omega = 8\pi a I_{0\omega}\left(\sqrt{\frac{\varepsilon}{\mu}} + \sqrt{\frac{\varepsilon}{\pi}}\right)\int_0^\chi dy \int_0^{\pi/2} \frac{(\cos^4\theta + \cos^2\theta + y^2\sin^2\theta)\cos^4\theta \, d\theta}{\left|\sqrt{\frac{\varepsilon}{\mu}}\cos^3\theta + y^2\sin^2\theta\right|^2} .$$

Since y varies from 0 to $\chi \sim 1$, then with the same degree of precision with which our estimate is carried out we can simply neglect terms in y^2. We thereby obtain, using (IV.29) again,

$$P_\omega = 6\pi^2\chi a \frac{\sqrt{\mu}(\sqrt{\varepsilon} + \sqrt{\varepsilon^*})}{|\varepsilon|} I_{0\omega} = 6\pi^2\chi ak\mu d I_{0\omega}. \qquad \text{(IV.38)}$$

In the region of validity of the Rayleigh-Jeans law and per unit of cylinder surface, this gives

$$p_\omega = \frac{P_\omega}{2\pi a} = \frac{3\chi\theta}{4\pi^2} k^3 \mu d .$$

Further on, in another problem we shall be able to carry out a much finer evaluation (section 15), which shows that the coefficient is equal

to $2/3\pi^2$ (i.e., $\chi = 8/9$). Thus, the proper value of power, emitted from a unit surface of a well conducting surface, is

$$P_\omega = \frac{2\theta}{3\pi^2} k^3 \mu d . \qquad \text{(IV.39)}$$

Thin, poorly conducting cylinder

$$ka \ll 1, \qquad a \ll d,$$

i.e., $\lambda_0 a \ll 1$ and $|\lambda| a \ll 1$. Beside formulae (IV.33), we can in this case utilize the first terms of power series for Bessel functions as well

$$J_0(\lambda a) = 1, \quad J_n(\lambda a) = \frac{1}{n!}\left(\frac{\lambda a}{2}\right)^n, \quad J_0'(\lambda a) = -\frac{\lambda^2 a}{2},$$

$$J_n'(\lambda a) = \frac{1}{(n-1)!a}\left(\frac{\lambda a}{2}\right)^m .$$

In view of the fact that expression (IV.26) here, too, is of the order $a^{2|n|}$, we shall again limit ourselves to the zero term of the sum (in this way, the symmetrical zero wave always is dominant for a thin cylinder). For $n = 0$, we have

$$\lambda^2 JJ'^* - \lambda^{*2} J^* J' \approx 0, \qquad JJ'^* - J^* J' = \frac{\lambda^2 - \lambda^{*2}}{2} a,$$

$$\Delta_1 = \Delta_2 = -\frac{2i}{\pi a}, \qquad \delta = 0, \qquad \Delta = \Delta_1 \Delta_2 .$$

Finally,

$$A_{P_0}(h) + A_{Q_0}(h) = \frac{i\pi a^2 \lambda_0^2 (\lambda^2 - \lambda^{*2})}{2k^2} = \pi a^2 \varepsilon'' (k^2 - h^2).$$

Substituting this into (IV.27), we get

$$P_\omega = \frac{2\pi^2 a^2 \varepsilon''}{k^2} I_{0\omega} \int_0^k (k^2 - h^2) dh = \frac{4\pi^2 a^2 \varepsilon''}{3} I_{0\omega} . \qquad \text{(IV.40)}$$

For $I_{0\alpha} = \frac{\varepsilon k^2}{4\pi^3}$, the power emitted from a unit cylinder surface is

$$P_\omega = \frac{P_\omega}{2\pi a} = \frac{\theta k^3 a \varepsilon''}{6\pi^2} . \qquad \text{(IV.41)}$$

V. Derivation of formulae (14.7), (14.8) and (14.9)

Substitution of formula (14.4), where $\vec{N}$ and $\vec{M}$ are expressed by formulae (14.1), into condition (14.5) gives

$$\sum_{n=0}^{\infty} \sum_{m=-n}^{+n} \left\{ A_{p_n} \frac{imP}{\sin\theta} + B_{q_n} P' \right\} e^{im\varphi} = -\sqrt{\varepsilon}\,\mathcal{K}_\theta$$

$$\sum_{n=0}^{\infty} \sum_{m=-n}^{+n} \left\{ -A_{p_n} P' + B_{q_n} \frac{imP}{\sin\theta} \right\} e^{im\varphi} = -\sqrt{\varepsilon}\,\mathcal{K}_\varphi \qquad \text{(V.1)}$$

where $P = P_n^{|m|}(\cos\theta)$, $P' = \frac{dP}{d\theta}$ and where the following notations have been introduced:

$$p_n = \sqrt{\varepsilon}\, z + \frac{i\sqrt{\mu}}{\alpha} \frac{d\alpha z}{d\alpha}$$

$$q_n = \frac{\sqrt{\varepsilon}}{\alpha} \frac{d\alpha z}{d\alpha} - i\sqrt{\mu}\, z . \qquad \text{(V.2)}$$

Let us now utilize the fact that

$$\int_0^{\pi} \left\{ \frac{dP_n^{|m|}}{d\theta} \frac{dP_{n_1}^{|m|}}{d\theta} + \frac{m^2}{\sin^2\theta} P_n^{|m|} P_{n_1}^{|m|} \right\} \sin\theta \, d\theta = \delta_{nn_1} \gamma_{nm} ,$$

$$\gamma_{nm} = \frac{2n(n+1)}{2n+1} \frac{(n+|m|)!}{(n-|m|)!} , \qquad \int_{-\pi}^{+\pi} e^{i(m-m_1)\varphi}\, d\varphi = 2\pi\delta_{mm_1} ,$$

$$\int_0^{\pi} \frac{m}{\sin\theta} \left\{ \frac{dP_n^{|m|}}{d\theta} P_{n_1}^{|m|} + P_n^{|m|} \frac{dP_{n_1}^{|m|}}{d\theta} \right\} \sin\theta \, d\theta = 0 . \qquad \text{(V.3)}$$

Therefore, if we multiply the first equation of (V.1) by

$- imP_{n_1}^{|m|}\, e^{-im_1\varphi}\, d\theta\, d\varphi$, the second by $-\dfrac{dP_{n_1}^{|m|}}{d\theta}\, e^{-im_1\varphi} \sin\theta\, d\theta\, d\varphi$,

add them, and integrate over the unit sphere, and then carry out the analogous operation, but multiplying now the first equation of (V.1) by

$\dfrac{dP_{n_1}^{|m|}}{d\theta}\, e^{-im_1\varphi} \sin\theta\, d\theta\, d\varphi$ and the second by $- imP_{n_1}^{|m|}\, e^{-im_1\varphi}\, d\theta\, d\varphi$,

we obtain

$$A = \frac{\sqrt{\varepsilon}}{2\pi p_n \gamma_{nm}} \int_{-\pi}^{+\pi} e^{-im\varphi}\, d\varphi \int_0^{\pi} \left(\frac{imP}{\sin\theta} \mathcal{H}_\theta + P' \mathcal{H}_\varphi \right) \sin\theta\, d\theta\,, \tag{V.4}$$

$$B = -\frac{\sqrt{\varepsilon}}{2\pi q_n \gamma_{nm}} \int_{-\pi}^{+\pi} e^{-im\varphi}\, d\varphi \int_0^{\pi} \left(P' \mathcal{H}_\theta - \frac{imP}{\sin\theta} \mathcal{H}_\varphi \right) \sin\theta\, d\theta\,.$$

Averaging the squares of the moduli of the coefficients of A and B by means of the correlation function (14.6) and formula (V.3) we find

$$\overline{|A|^2} = \frac{\mathcal{C}\,|\varepsilon|}{2\pi a^2 \gamma_{nm} |p_n|^2}, \qquad \overline{|B|^2} = \frac{\mathcal{C}\,|\varepsilon|}{2\pi a^2 \gamma_{nm} |q_n|^2}. \tag{V.5}$$

The energy flow through a sphere of radius R is

$$P_\omega = R^2 \int_{-\pi}^{+\pi} d\varphi \int_0^{\pi} S_{\omega R} \sin\theta\, d\theta = \frac{cR^2}{4\pi} \int_{-\pi}^{+\pi} d\varphi \int_0^{\pi} (\overline{E_\theta H_\varphi^*} - \overline{E_\varphi H_\theta^*} +$$

$$+ \overline{E_\theta^* H_\varphi} - \overline{E_\varphi^* H_\theta}) \sin\theta\, d\theta\,.$$

Introducing here (14.4) and using again formulae (V.3), we bring this expression to the form $(\gamma_{0m} = 0)$

$$P_\omega = \frac{icR^2}{2} \sum_{n=1}^{\infty} \sum_{m=-n}^{+n} (s^*s' - ss'^*) \gamma_{nm} \left(\overline{|A|^2} + \overline{|B|^2} \right) .$$

Furthermore[1]

$$s^*s' - ss'^* = s_n^*(\rho) \frac{ds_n(\rho)}{d\rho} - s_n(\rho) \frac{ds_n^*(\rho)}{d\rho} =$$

$$= \frac{n(s_{n-1} s_n^* - s_{n-1}^* s_n) + (n+1)(s_n s_{n+1}^* - s_n^* s_{n+1})}{2n+1}$$

and, since

$$s_{n-1} s_n^* - s_{n-1}^* s_n = \frac{i\pi}{\rho} \left(J_{n-\frac{1}{2}} N_{n+\frac{1}{2}} - J_{n+\frac{1}{2}} N_{n-\frac{1}{2}} \right) = -\frac{2i}{\rho^2}$$

we obtain

$$s^*s' - ss'^* = -\frac{2i}{\rho^2} = -\frac{2i}{k^2R^2} . \qquad \text{(V.6)}$$

The expression for P_ω finally assumes the form (14.7).

Let us substitute into (14.7) the mean squares of the moduli of A and B from (V.5), and replace $\mathcal{C}$ by formula (12.6) and express $|\varepsilon|$ in terms of the skin-layer thickness d by means of (12.14):

$$\sqrt{\varepsilon} = \frac{1-i}{kd\sqrt{\mu}}, \qquad |\varepsilon| = \frac{2}{k^2d^2\mu} .$$

This gives

$$P_\omega = \frac{8\pi^2 I_0 \omega}{k^5 a^2 d} \sum_{n=1}^{\infty} \sum_{m=-n}^{+n} \left(\frac{1}{|p_n|^2} + \frac{2}{|q_n|^2} \right) . \qquad \text{(V.7)}$$

Since p_n and q_n do not depend on m [see (V.2)], summation with respect to m gives the multiplier (2n + 1) and we obtain formula (14.8).

1. J. A. Stratton, Theory of Electromagnetism, p. 377 (M., 1948).

It is not difficult to obtain the attenuation coefficients of the H_{nm}- and E_{nm}- waves with the help of the boundary conditions (11.6), i.e., the uniform conditions (14.5)

$$\begin{aligned} \sqrt{\varepsilon}\, E_\theta + \sqrt{\mu}\, H_\varphi &= 0, \\ \sqrt{\varepsilon}\, E_\varphi - \sqrt{\mu}\, H_\theta &= 0 \end{aligned} \qquad (\rho = \alpha) \qquad (V.8)$$

where by E and H are meant the summed potentials of the incident and reflected waves. In order to obtain contracting (incident) waves instead of propagating (reflected) waves, it is sufficient to replace in (14.1) $z = \sqrt{\frac{\pi}{2\rho}}\, H^{(2)}_{n+\frac{1}{2}}(\rho)$ by $z^* = \sqrt{\frac{\pi}{2\rho}}\, H^{(1)}_{n+\frac{1}{2}}(\rho)$. Thus, in the incident H_{nm}- wave we have

$$E_\theta = \frac{imP}{\sin\theta} z^* e^{im\varphi}, \quad E_\varphi = -P' z^* e^{im\varphi}, \quad H_\theta = \frac{iP'}{\rho}\frac{d\rho z^*}{d\rho} e^{im\varphi},$$

$$H_\varphi = -\frac{mP}{\rho\sin\theta}\frac{d\rho z^*}{d\rho} e^{im\varphi},$$

and in the reflected wave similar expressions with z instead of z^* and with some amplitude R_{nm}. In contrast to the previously examined problem concerning cylinder radiation, here, in view of the high conductivity, the E and H waves upon reflection do not get mixed up, i.e., H_{nm}- waves upon reflection give also H_{nm}- waves. It is understood that this takes place with the approximation within which conditions (V.8) are valid.

Upon substituting the summed strengths into the boundary conditions (V.8), we obtain from both conditions one and the same expression for the amplitude of the reflected wave, namely

$$R_{nm} = \frac{\sqrt{\varepsilon}\,\alpha z^* + i\sqrt{\mu}(\alpha z^*)'}{\sqrt{\varepsilon}\,\alpha z + i\sqrt{\mu}(\alpha z)'} .$$

Consequently, the attenuation coefficient of H_{nm} waves will be

$$A_n = 1 - |R_{nm}|^2 = i\sqrt{\mu}\,(\sqrt{\varepsilon^*} + \sqrt{\varepsilon}\,)\,\alpha\,\frac{z^*(\alpha z)' - z(\alpha z^*)'}{|\sqrt{\varepsilon}\,\alpha z + i\sqrt{\mu}\,(\alpha z)'|^2} =$$

$$= \frac{i\sqrt{\mu}\,(\sqrt{\varepsilon} + \sqrt{\varepsilon^*})}{|p_n|^2}\,(z^* z' - z z'^*) = \frac{4}{k^3 ad |p_n|^2},$$

where we have utilized formulas (V.2), (V.6) and (V.7). The index m has been omitted from the attenuation coefficient, since the latter is independent of m.

An analogous evaluation for the E_{nm} - waves gives

$$A_n' = \frac{4}{k^3 ad |q_n|^2}$$

so that

$$\frac{1}{|p_n|^2} + \frac{1}{|q_n|^2} = \frac{k^3 ad}{4}\,(A_n + A_n')\,.$$

Substituting this into (14.8), we obtain formula (14.9).

VI. Derivation of formulas (15.7), (15.12) and (15.17)

For the z-components of the density of energy flow we have

$$S_{\omega z} = \frac{c}{4\pi}\,(\overline{E_x H_y^*} - \overline{E_y H_x^*} + \overline{E_x^* H_y} - \overline{E_y^* H_x}).$$

Substituting in here from (15.6) the expression for the transverse components of E and H, we find

$$S_{\omega z} = \frac{c}{4\pi}\sum_{m,n}\sum_{\mu,\nu}\left\{\overline{A_{mn}A^*_{\mu\nu}}\,\frac{\beta_{mn} k}{\chi^2_{mn}\chi^2_{\mu\nu}}\,e^{i(\beta^*_{\mu\nu} - \beta_{mn})z}\left[\frac{\partial\varphi_{mn}}{\partial x}\frac{\partial\varphi^*_{\mu\nu}}{\partial x} + \frac{\partial\varphi_{mn}}{\partial y}\frac{\partial\varphi^*_{\mu\nu}}{\partial y}\right] +\right.$$

$$+ B_{mn} B^{*}_{\mu\nu} \frac{\beta'^{*}_{\mu\nu} k}{\varkappa'^{2}_{mn} \varkappa'^{2}_{\mu\nu}} e^{i(\beta'^{*}_{\mu\nu} - \beta'_{mn})z} \left[\frac{\partial \Psi_{mn}}{\partial x} \frac{\partial \Psi^{*}_{\mu\nu}}{\partial x} + \frac{\partial \Psi_{mn}}{\partial y} \frac{\partial \Psi^{*}_{\mu\nu}}{\partial y} \right] -$$

$$- B_{mn} A^{*}_{\mu\nu} \frac{k^2}{\varkappa'^{2}_{mn} \varkappa^{2}_{\mu\nu}} e^{i(\beta^{*}_{\mu\nu} - \beta'_{mn})z} \left[\frac{\partial \Psi_{mn}}{\partial x} \frac{\partial \Phi^{*}_{\mu\nu}}{\partial y} - \frac{\partial \Psi_{mn}}{\partial y} \frac{\partial \Phi^{*}_{\mu\nu}}{\partial x} \right] -$$

$$- A_{mn} B^{*}_{\mu\nu} \frac{\beta_{mn} \beta'^{*}_{\mu\nu}}{\varkappa^{2}_{mn} \varkappa'^{2}_{\mu\nu}} e^{i(\beta'^{*}_{\mu\nu} - \beta_{mn})z} \left[\frac{\partial \Phi_{mn}}{\partial y} \frac{\partial \Psi^{*}_{\mu\nu}}{\partial x} - \frac{\partial \Phi_{mn}}{\partial x} \frac{\partial \Psi^{*}_{\mu\nu}}{\partial y} \right] +$$

$$+ \text{complex conjugate} \} .$$

To obtain the total energy flow we must multiply S_z by $dS = dxdy$ and integrate over the cross-sectional area S of the waveguide. Here, by virtue of (15.1), (15.3) and (15.5) we have

$$\int_S \left[\frac{\partial \Phi_{mn}}{\partial x} \frac{\partial \Phi^{*}_{\mu\nu}}{\partial x} + \frac{\partial \Phi_{mn}}{\partial y} \frac{\partial \Phi^{*}_{\mu\nu}}{\partial y} \right] dxdy =$$

$$= - \int_S \Phi_{mn} \nabla^2 \Phi^{*}_{\mu\nu} dxdy = \varkappa^{2}_{\mu\nu} \int_S \Phi_{mn} \Phi^{*}_{\mu\nu} dxdy = \varkappa^{2}_{\mu\nu} S \delta_{m\mu} \delta_{n\nu} \qquad \text{(VI.1)}$$

and analogously for the bracket with Ψ. The integrals of the brackets, containing Φ and Ψ, give zero

$$\int_S \left[\frac{\partial \Psi_{mn}}{\partial x} \frac{\partial \Phi^{*}_{\mu\nu}}{\partial y} - \frac{\partial \Psi_{mn}}{\partial y} \frac{\partial \Phi^{*}_{\mu\nu}}{\partial x} \right] dxdy =$$

$$= \int_S \Phi^{*}_{\mu\nu} \left(\frac{\partial^2 \Psi_{mn}}{\partial x \partial y} - \frac{\partial^2 \Psi_{mn}}{\partial x \partial y} \right) dxdy = 0 . \qquad \text{(VI.2)}$$

Finally

$$P_\omega = \int_S S_{\omega z}\,dxdy = \frac{kcS}{4\pi}\sum_{m,n}\left\{\frac{\beta+\beta^*}{\chi^2}\overline{|A|^2}\,e^{i(\beta^*-\beta)z} + \right.$$

$$\left. + \frac{\beta'+\beta'^*}{\chi'^2}\overline{|B|^2}\,e^{i(\beta'^*-\beta')z}\right\}.$$

Since the corresponding values of χ_{mn} and χ'_{mn} are real, the quantities β_{mn} and β'_{mn} are either real or imaginary. In the latter case $\beta + \beta^* = 0$ and therefore only terms with real values for β ("sub-critical" non-extinguishing waves) will remain in P_ω. This gives formula (15.7).

Substituting (15.6) into the boundary conditions (15.8) and introducing temporarily the notation

$$\tilde{A} = \frac{k\sqrt{\mu} + \beta\sqrt{\varepsilon}}{i\chi^2}A, \quad \tilde{B} = \frac{\beta'\sqrt{\mu} + k\sqrt{\varepsilon}}{i\chi'^2}, \tag{VI.3}$$

we obtain

$$\sum_{m,n}\left(\tilde{A}\frac{\partial\Phi}{\partial x} + \tilde{B}\frac{\partial\Psi}{\partial y}\right) = -\sqrt{\varepsilon}\,\mathcal{H}_x$$

$$\sum_{m,n}\left(\tilde{A}\frac{\partial\Phi}{\partial y} - \tilde{B}\frac{\partial\Psi}{\partial x}\right) = -\sqrt{\varepsilon}\,\mathcal{H}_y\,. \tag{VI.4}$$

After multiplying the first equation of (VI.4) by $\frac{\partial\Phi^*_{\mu\nu}}{\partial x}\,dxdy$, and the second by $\frac{\partial\Phi^*_{\mu\nu}}{\partial y}\,dxdy$, let us add them and integrate over S. By virtue of (VI.1) and (VI.2), we obtain

$$\tilde{A}_{mn}\chi^2_{mn}S = -\sqrt{\varepsilon}\int_S\left(\mathcal{H}_x\frac{\partial\Phi^*_{mn}}{\partial x} + \mathcal{H}_y\frac{\partial\Phi^*_{mn}}{\partial y}\right)dxdy\,.$$

Performing an operation analogous to (VI.4) but with multiplication of the

first equation by $\frac{\partial \Phi^{*}_{\mu\nu}}{\partial y}$ dxdy and of the second by $\frac{\partial \Phi^{*}_{\mu\nu}}{\partial x}$ dxdy we get:

$$\tilde{B}_{mn} \varkappa^{2}_{mn} S = \sqrt{\varepsilon} \int_{S} \left(\mathcal{H}_y \frac{\partial \Phi^{*}_{mn}}{\partial x} - \mathcal{H}_x \frac{\partial \Phi^{*}_{mn}}{\partial y} \right) dxdy \ .$$

The evaluation of the average square of the modulus, for instance, A_{mn}, which can be done with the help of the correlation function (15.9), gives

$$\varkappa^{4}\overline{|\tilde{A}|^{2}}S^{2} = |\varepsilon| \mathcal{C} \iint_{SS} dxdy\, dx'dy' \delta(x - x') \delta(y - y') \left[\frac{\partial \Phi^{*}}{\partial y'} \frac{\partial \Phi}{\partial y} + \right.$$

$$\left. + \frac{\partial \Phi^{*}}{\partial x'} \frac{\partial \Phi}{\partial x} \right] = |\varepsilon| \mathcal{C} \int_{S} dxdy \left[\frac{\partial \Phi}{\partial y} \frac{\partial \Phi^{*}}{\partial y} + \frac{\partial \Phi}{\partial x} \frac{\partial \Phi^{*}}{\partial x} \right] =$$

$$= |\varepsilon| \mathcal{C} S \varkappa^{2} , \qquad \text{(VI.5)}$$

where the last expression follows from (VI.1). The expression for $\overline{|\tilde{B}|^{2}}$ is calculated analogously, after which with the help of (VI.3) we obtain formula (15.10).

Let us note that $\overline{|\tilde{A}|^{2}}$ and $\overline{|\tilde{B}|^{2}}$ for the case of non-uniform distribution of the conductance (this means, also of the lateral field) over the partition can be found in the same manner. Let this distribution be given by the function $\varphi(x, y)$, which must consequently be introduced as a multiplier in the right hand sides of equations (15.8) and (VI.4). Then, as one can easily convince oneself, instead of formula (VI.5) for $\overline{|A|^{2}}$ we get

$$\varkappa^{4}\overline{|A|^{2}}S^{2} = |\varepsilon| \mathcal{C} \int_{S} \varphi^{2}(x, y)\, dxdy \left[\frac{\partial \Phi}{\partial y} \frac{\partial \Phi^{*}}{\partial y} + \frac{\partial \Phi}{\partial x} \frac{\partial \Phi^{*}}{\partial x} \right]$$

and analogously for $\overline{|\tilde{B}|^{2}}$.

Let us limit ourselves to the derivation of the attenuation coefficient only for E_{om} - waves. Since we are talking about fixed m and n, we shall not write these indices for $\varkappa$, and Φ. Let the indicated wave,

propagating in the negative direction along the z-axis, be incident on the partition, placed in the plane $z = 0$. From (15.6), we have (changing the sign on β)

$$E_x = -\frac{\beta}{i\varkappa^2}\frac{\partial\Phi}{\partial x}e^{i\beta z}, \qquad E_y = -\frac{\beta}{i\varkappa^2}\frac{\partial\Phi}{\partial y}e^{i\beta z},$$

$$H_x = -\frac{k}{i\varkappa^2}\frac{\partial\Phi}{\partial y}e^{i\beta z}, \qquad H_y = \frac{k}{i\varkappa^2}\frac{\partial\Phi}{\partial x}e^{i\beta z}.$$

For the reflected wave the expressions for the strengths (E^r or H^r) will differ in the sign of β [i.e., the sign will be the same as in (15.6)] and by some amplitude constant A^r.

Each of the boundary conditions

$$\begin{aligned} \sqrt{\mu}\,(H_x + H_x^r) &= \sqrt{\varepsilon}\,(E_y + E_y^r) \\ \sqrt{\mu}\,(H_y + H_y^r) &= -\sqrt{\varepsilon}\,(E_x + E_x^r) \end{aligned} \qquad (z = 0)$$

gives one and the same result, namely

$$A^r(\beta\sqrt{\varepsilon} + k\sqrt{\mu}) = \beta\sqrt{\varepsilon} - k\sqrt{\mu}.$$

The powers, carried by the incident and reflected waves, are in a ratio like the squares of the amplitude moduli, i.e., the reflection coefficient is

$$R = |A^r|^2 = \left|\frac{\beta\sqrt{\varepsilon} - k\sqrt{\mu}}{\beta\sqrt{\varepsilon} + k\sqrt{\mu}}\right|^2$$

and the attenuation coefficient is

$$A = 1 - R = \frac{2k\beta(\sqrt{\varepsilon^*\mu} + \sqrt{\varepsilon\mu})}{|\beta\sqrt{\varepsilon} + k\sqrt{\mu}|^2}.$$

VII. Derivation of formulae (16.9) and (16.13)

In a rectangular waveguide $\varkappa_{mn}$ and $\varkappa'_{mn}$ are equal and expressed by formula (15.11). Summation in (16.8) is carried out over all m and n for which $\varkappa_{mn} > k$, so that the passage to the integral is of the form

$$\sum_{m,n}{}'' \longrightarrow \iint\limits_{\varkappa_{mn}=k}^{\infty} dm\,dn \;.$$

Let us introduce the variables ξ and φ

$$\frac{\lambda m}{2a} = \sqrt{1 + \xi^2}\cos\varphi, \qquad \frac{\lambda n}{2b} = \sqrt{1 + \xi^2}\sin\varphi,$$

in which

$$\varkappa^2 = k^2(1 + \xi^2), \qquad \alpha = k\xi, \qquad dm\,dn = \frac{k^2 ab}{\pi^2}\xi\,d\xi\,d\varphi$$

(integration with respect to φ from 0 to $\pi/2$, with respect to ξ from 0 to ∞). If we then introduce into (16.8) the parameter δ, which is determined by formula (14.10)

$$\delta = k\mu d = (1 - i)\sqrt{\frac{\mu}{\varepsilon}}, \qquad (\delta \ll 1) \tag{VII.1}$$

and substitute the value for $\mathcal{C}$

$$\mathcal{C} = \frac{2\pi^2\mu d}{k}u_{0\omega} = \frac{2\pi^2\delta}{k^2}u_{0\omega}, \tag{VII.2}$$

we obtain ($S = ab$)

$$\begin{aligned} w_{\omega\,\mathrm{quas}} &= u_{0\omega}\frac{S\delta}{2}\int_0^{\infty} e^{-2kz\xi}(1 + \xi^2)\xi\left\{\left|\frac{\delta}{1 - i} - i\xi\right|^{-2} + \left|\frac{i\delta\xi}{1 - i} - i\right|^{-2}\right\}d\xi = \\ &= u_{0\omega}\frac{S\delta}{2}\int_0^{\infty} e^{-2kz\xi}(1 + \xi^2)\left\{\frac{1}{\xi^2 - \xi\delta + \frac{\delta^2}{2}} + \frac{1}{\frac{\xi^2\delta^2}{2} + \xi\delta + 1}\right\}d\xi. \end{aligned} \tag{VII.3}$$

This formula coincides with the previously derived formula in Appendix I for the energy density of the quasi-stationary field in the case of a semi-infinite well conducting medium [see (1.24)]. Understandably, the linear density $w_{\omega\,quas}$ is related to the volume density averaged over the cross-section $u_{\omega\,quas}$ by the expression $w_{\omega\,quas} = u_{\omega\,quas}\, S$.

For the cases $ks \gg 1/\delta$ and $ks \ll \delta$ formula (1.23) is valid and has been adduced in the basic text under the number (16.9).

Averaging of (16.10) over $\xi = \beta\ell$ and $\xi' = \beta'\ell$ according to formula (7.10) and substitution of the value (VII.2) for $\mathcal{C}$ give

$$U_{\omega} = \frac{\pi\ell}{k} u_{0\omega} \sum_{m,n}{}' \left(\frac{1}{\beta_{mn}} + \frac{1}{\beta'_{mn}} \right) .$$

For a rectangular waveguide, according to (15.14), we have

$$\beta_{mn} = \beta'_{mn} = k\sqrt{1 - \frac{\lambda^2}{4}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)}$$

Therefore, upon passing from summation to integration

$$\sum_{m,n}{}' \longrightarrow \int\!\!\int_0^{\varkappa_{mn} = k} dm\, dn$$

and changing variables to θ and φ

$$\frac{\lambda m}{2a} = \sin\theta\cos\varphi, \qquad \frac{\lambda n}{2b} = \sin\theta\sin\varphi$$

(θ from $\pi/2$ to 0, φ from 0 to $\pi/2$), in which

$$\beta = \beta' = k\cos\theta, \qquad dm\, dn = \frac{k^2 ab}{\pi^2}\cos\theta\sin\theta\, d\theta\, d\varphi ,$$

we obtain (16.13)

$$U_{\omega} = \frac{\pi\ell}{k} u_{0\omega} \int_{\pi/2}^{0} \frac{k^2 S}{\pi^2}\cos\theta\sin\theta\, d\theta \cdot \frac{2}{k\cos\theta}\int_0^{\pi/2} d\varphi = \ell S \cdot u_{0\omega} .$$

VIII. Derivation of formula (23.9)

For the evaluation of the intensity according to formula (21.19)

$$I_0 = \frac{I_{0\omega}}{2k^2} \left| \frac{F_0 \sqrt{F_0^2 + F_0'^2} \sin\theta}{\frac{\partial}{\partial\theta}\left\{ \frac{F_0 \cos + F_0' \sin}{\sqrt{F_0^2 + F_0'^2}} \right\}} \right| , \qquad \text{(VIII.1)}$$

it is necessary first of all to find F_0' . For this reason we shall differentiate with respect to θ the scattering equation (23.1) itself. This gives

$$F_0' = - \frac{F_0 v \sin\theta \cos\theta}{u + v \sin^2\theta} \qquad \text{(VIII.2)}$$

where

$$u = [(\alpha_1 - \alpha_3)\sin^2\theta + 2\alpha_3] F_0^2 - 2\alpha_1\alpha_3 ,$$

$$v = (\alpha_1 - \alpha_3)(F_0^2 - \alpha_1) + \beta^2 . \qquad \text{(VIII.3)}$$

With the help of (VIII.2), we find

$$F_0 \cos\theta + F_0' \sin\theta = \frac{F_0 u \cos\theta}{u + v \sin^2\theta} ,$$

$$\text{(VIII.4)}$$

$$\sqrt{F_0^2 + F_0'^2} = \frac{F_0 \sqrt{u^2 + 2uv \sin^2\theta + v^2 \sin^2\theta}}{u + v \sin^2\theta}$$

whence

$$\frac{F_0 \cos\theta + F_0' \sin\theta}{\sqrt{F_0^2 + F_0'^2}} = \frac{u \cos\theta}{\sqrt{u^2 + 2uv \sin^2\theta + v^2 \sin^2\theta}}$$

furthermore

$$\frac{\partial}{\partial\theta}\left\{\frac{F_0\cos\theta + F_0'\sin\theta}{\sqrt{F_0^2 + F_0'^2}}\right\} =$$

$$= \frac{(u + v)[(vu' - v'u)\sin\theta\cos\theta - u(u + v)]\sin\theta}{(u^2 + 2uv\sin^2\theta + v^2\sin^2\theta)^{3/2}} . \qquad \text{(VIII.5)}$$

Substituting (VIII.4) and (VIII.5) into (VIII.1), we obtain

$$I_0 = \frac{I_{\nu\omega}F_0^2}{2k^2} \times$$

$$\times \frac{(u^2 + 2uv\sin^2\theta + v^2\sin^2\theta)^2}{|(u + v)(u + v\sin^2\theta)[(vu' - v'u)\sin\theta\cos\theta - u(u + v)]|} . \qquad \text{(VIII.6)}$$

In accordance with (VIII.3) and (VIII.2), we have then

$$u' = \frac{2F_0^2\,[(\alpha_1 - \alpha_3)u - 2\alpha_3 v]\sin\theta\cos\theta}{u + v\sin^2\theta} ,$$

$$v' = \frac{2F_0^2\,v(\alpha_1 - \alpha_3)v\sin\theta\cos\theta}{u + v\sin^2\theta} ,$$

so that

$$vu' - v'u = \frac{4F_0^2\,v[(\alpha_1 - \alpha_3)u - \alpha_3 v]\sin\theta\cos\theta}{u + v\sin^2\theta} .$$

Introducing this into (VIII.6), we obtain for I_0 an expression no longer containing derivatives of θ

$$I_0 = \frac{I_0 \omega P_0^2}{2k^2} \times \tag{VIII.7}$$

$$\times \frac{(u^2 + 2uv \sin^2\theta + v^2 \sin^2\theta)^2}{\left|(u + v)\left\{4P_0^2 v[(\alpha_1 - \alpha_3)u - \alpha_3 v]\sin^2\theta\cos^2\theta - u(u + v)(u + v\sin^2\theta\right\}\right|}$$

For further manipulation of this form we must use the explicit expression (23.8) for the root P_0^2. Substituting it into (VIII.3), we find

$$u = \frac{B \sin^2\theta + D}{2A}, \quad v = -\frac{B}{2A}, \tag{VIII.8}$$

where A, B and D are expressed by formulae (23.10). From (VIII.8) and (23.10) it follows that

$$(\alpha_1 - \alpha_3)u - \alpha_3 v = \frac{C}{2}, \quad u + v = \frac{D - B\cos^2\theta}{2A},$$

$$u + v\sin^2\theta = E, \tag{VIII.9}$$

$$u^2 + 2uv\sin^2\theta + v^2\sin^2\theta = \frac{D^2 + B^2\sin^2\theta\cos^2\theta}{4A^2},$$

where C and E, too, have values given by (23.10). Substituting (VIII.8) and (VIII.9) into (VIII.7), we obtain the first formula of (23.9).

To obtain the second formula of (23.9), i.e., the expression for the intensity I_e, we must utilize the second root of (23.8). Replacing P_0^2 by P_e^2 from (23.8) into (VIII.3), we get

$$u = \frac{C \sin^2\theta - D}{2A}, \quad v = -\frac{C}{2A} \tag{VIII.10}$$

and further

$$(\alpha_1 - \alpha_3)u - \alpha_3 v = \frac{B}{\varepsilon}, \qquad u + v = -\frac{D + C\cos^2\theta}{2A},$$

$$u + v\sin^2\theta = -\xi, \tag{VIII.10}$$

$$u^2 + 2uv\sin^2\theta + v^2\sin^2\theta = \frac{D^2 + C^2\sin^2\theta\cos^2\theta}{4A^2}.$$

Using the expressions for the quantities given above, the second formula of (23.9) is obtained from (VIII.7).

IX. Derivation of formula (27.18)

In the case of a small sphere ($\alpha = ka \ll 1$) the magnetic dipole wave H_{1m} ($m = -1, 0, 1$) is dominant, but initially we shall take the general case for the wave H_{nm}, for which according to (14.1) - (14.4),

$$E_R = 0, \qquad E = \frac{imz}{\sin\theta}Pe^{im\varphi}, \qquad E_\varphi = -zP'e^{im\varphi},$$

$$H_R = \frac{in(n+1)z}{\rho}Pe^{im\varphi}, \qquad H_\theta = \frac{i(\rho z)'}{\rho}P'e^{im\varphi}, \qquad H_\varphi = -\frac{m(\rho z)'}{\rho\sin\theta}Pe^{im\varphi}, \tag{IX.1}$$

where

$$z = \sqrt{\frac{\pi}{2\rho}}\,H^{(2)}_{n+\frac{1}{2}}(\rho), \qquad P = P_n^{|m|}(\cos\theta), \qquad \rho = kR$$

and the prime indicates the derivative of z with respect to ρ and of P with respect to θ.

For discrete harmonic oscillations

$$S_R = \frac{c}{16\pi}\left\{E_\theta H^*_\varphi - E_\varphi H^*_\theta + E^*_\theta H_\varphi - E^*_\varphi H_\theta\right\}. \tag{IX.2}$$

Therefore the emitted power is

$$P = \int S_R R^2 \sin\theta \, d\theta \, d\varphi =$$
$$= \frac{cR^2}{8} \int_0^{\pi} (E_\vartheta H_\varphi^* - E_\varphi H_\vartheta^* + E_\vartheta^* H_\varphi - E_\varphi^* H_\vartheta) \sin\vartheta \, d\theta \,. \tag{IX.3}$$

Joule's heat can be evaluated from the energy flow within the sphere, using the values in (IX.1) for the components of $\vec{H}$ and borrowing (for the accounting of a finite conductivity) the tangential components of $\vec{E}$ from the boundary conditions of M. A. Leontovich

$$\sqrt{\varepsilon}\, E_\vartheta = -\sqrt{\mu}\, H_\varphi, \quad \sqrt{\varepsilon}\, E_\varphi = \sqrt{\mu}\, H_\vartheta \,.$$

Substituting these expressions into (IX.2) and changing sign (i.e., taking the flow along the interior normal N to the sphere surface), we obtain the so-called surface density of Joule's heat

$$S_N = \frac{c(\sqrt{\varepsilon^*\mu} + \sqrt{\varepsilon\mu})}{16\pi|\varepsilon|} (|H_\varphi|^2 + |H_\theta|^2)_{\rho = \alpha} =$$
$$= \frac{c\delta}{16\pi} (|H_\varphi|^2 + |H_\vartheta|^2)_{\rho = \alpha} \,,$$

where $\delta = k\mu d$ (d is the skin-layer thickness). Therefore,

$$Q = \int S_N a^2 \sin\theta \, d\theta \, d\varphi = \frac{c\delta a^2}{8} \int_0^{\pi} (|H_\varphi|^2 + |H_\vartheta|^2)_{\rho=\alpha} \sin\vartheta \, d\theta \,. \tag{IX.4}$$

Finally, integrating the magnetic energy density $u_m = \frac{\vec{H}\vec{H}^*}{16\pi}$, we obtain U_m

$$U_m = \frac{1}{8k^3} \int_{\alpha}^{\infty} \rho^2 \, d\rho \int_0^{\pi} \left\{ |H_R|^2 + |H_\theta|^2 + |H_\varphi|^2 \right\} \sin\theta \, d\theta \,. \tag{IX.5}$$

In this formula it is essential to limit oneself to only the quasi-stationary terms of $\vec{H}$.

Substituting (IX.1) into (IX.3) - (IX.5) and using the following formulae

$$\int_0^{\pi}\left(P'^2 + \frac{m^2P^2}{\sin^2\theta}\right)\sin\theta\, d\theta = \gamma_{nm} = \frac{2n(n+1)}{2n+1}\frac{(n+|m|)!}{(n-|m|)!},$$

$$\int_0^{\pi} P^2 \sin\theta\, d\theta = \frac{\gamma_{nm}}{n(n+1)}, \quad z^*(\rho z)' - z(\rho z^*)' = \rho(z^*z' - zz'^*) = -\frac{2i}{\rho},$$

we find

$$P = \frac{c\gamma_{nm}}{4k^2}, \quad Q = \frac{c\delta\gamma_{nm}\,|(\alpha z)'|^2}{8k^2},$$

$$U_m = \frac{\gamma_{nm}}{8k^3}\int_0^{\infty}\left\{n(n+1)|z|^2 + |(\rho z)'|^2\right\} d\rho .$$

For $n = 1$ we have for the quasi-stationary field

$$z = \sqrt{\frac{\pi}{2\rho}}\, H^{(2)}_{3/2}(\rho) = -\frac{e^{-i\rho}}{\rho}\left(1 - \frac{i}{\rho}\right) \approx \frac{i}{\rho^2},$$

whence

$$(\rho z)' = -\frac{i}{\rho^2}.$$

Finally

$$P = \frac{c\gamma_{1m}}{4k^2}, \quad Q = \frac{c\delta\gamma_{1m}}{8k^2\alpha^4}, \quad U_m = \frac{\gamma_{1m}}{8k^3\alpha^3}.$$

Substitution of these expressions into (27.17) gives the following result independent of m

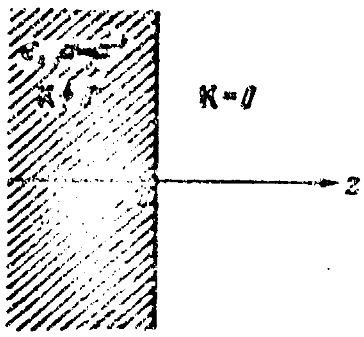


Fig. 1

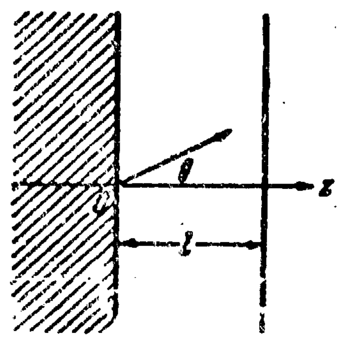


Fig. 2

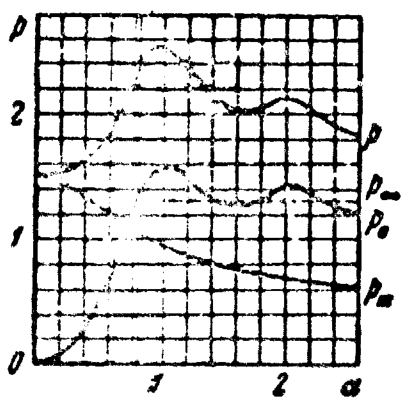


Fig. 3

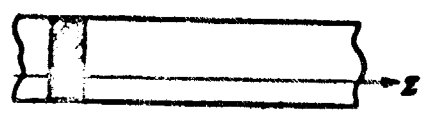


Fig. 4

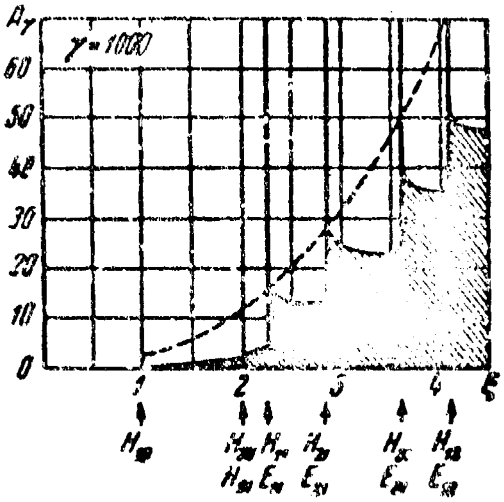


Fig. 5

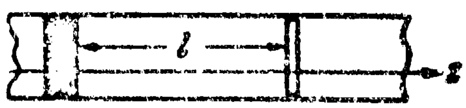


Fig. 6

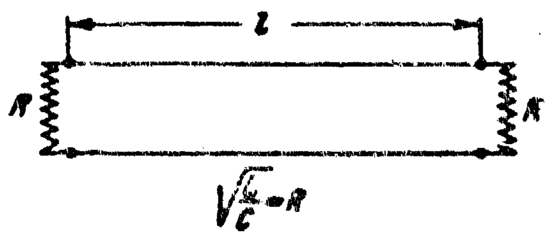


Fig. 7

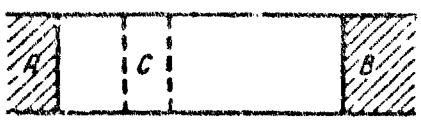
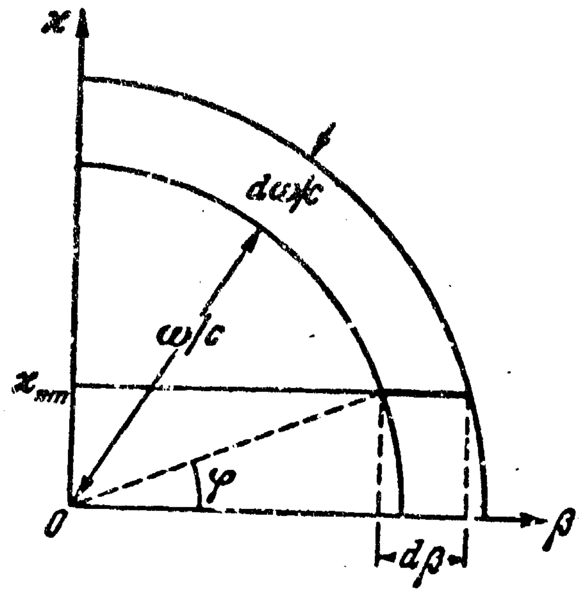
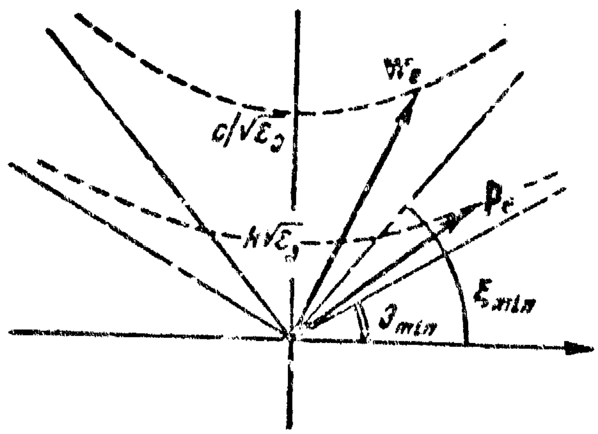


Fig. 8



$\varkappa$
$d\omega/c$
ω/c
$\varkappa_{min}$
φ
0
β
$d\beta$

Fig. 9



W_e
$c/\sqrt{\varepsilon_0}$
$h\sqrt{\varepsilon_0}$
p_e
ξ_{min}
ϑ_{min}

Fig. 10

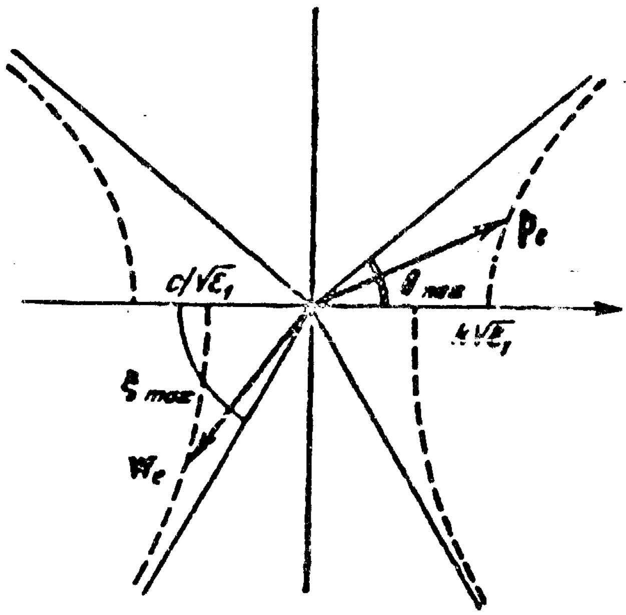
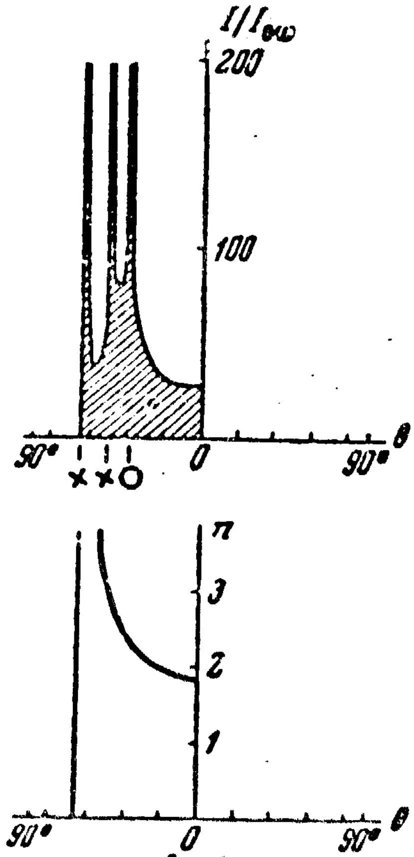


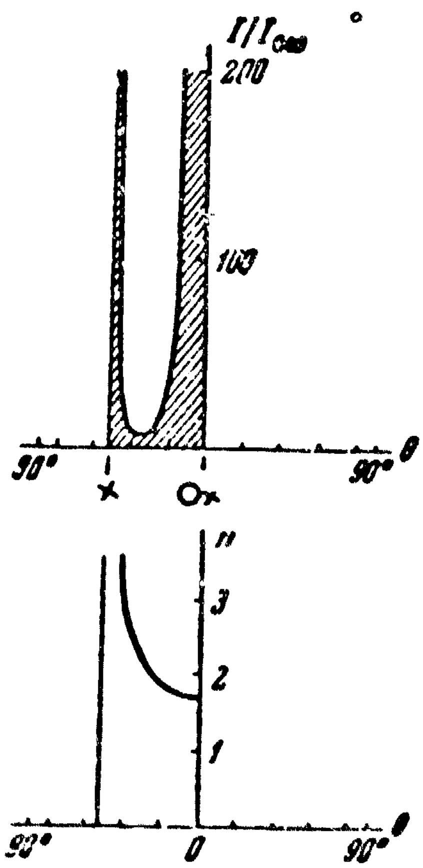
Fig. 11



$\zeta^2 = 1/18$

$$\left(\omega < \sqrt{\omega_0^2 + \frac{\omega_H^2}{2} - \omega_H\sqrt{\omega_0^2 + \frac{\omega_H^2}{4}}}\right)$$

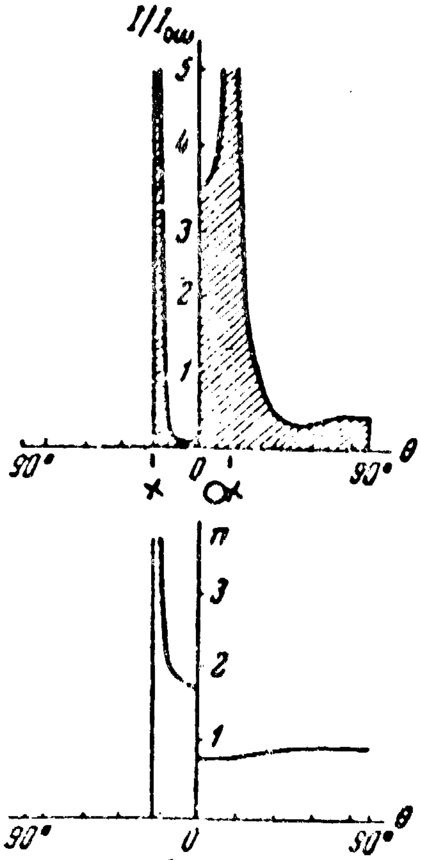
Fig. 12



$\zeta^2 = 1/9$

$$\left(\omega = \sqrt{\omega_0^2 + \frac{\omega_H^2}{2} - \omega_H\sqrt{\omega_0^2 + \frac{\omega_H^2}{4}}}\right)$$

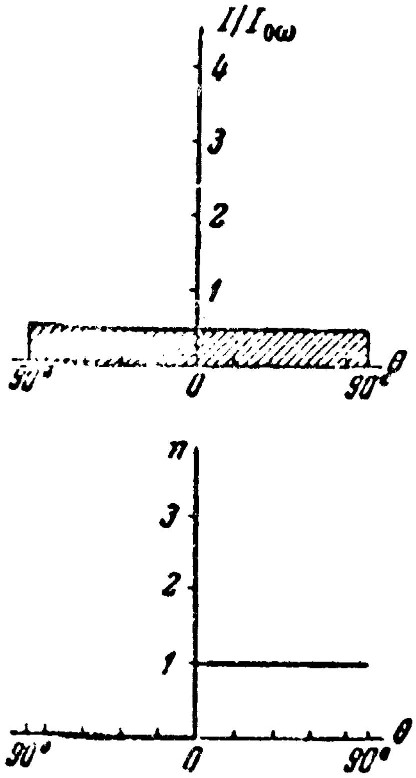
Fig. 13



$\zeta^2 = 1/3$

$\left(\sqrt{\omega_0^2 + \frac{\omega_H^2}{2} - \omega_H\sqrt{\omega_0^2 + \frac{\omega_H^2}{4}}} < \omega < \omega_0\right)$

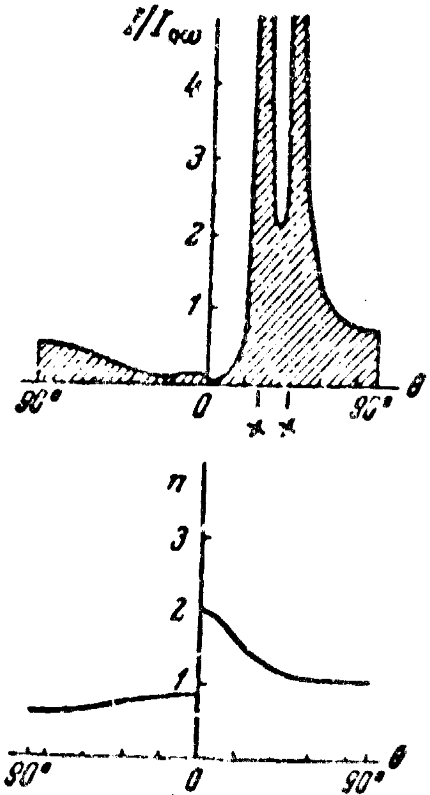
Fig. 14



$\zeta^2 = 4/9 + 0$

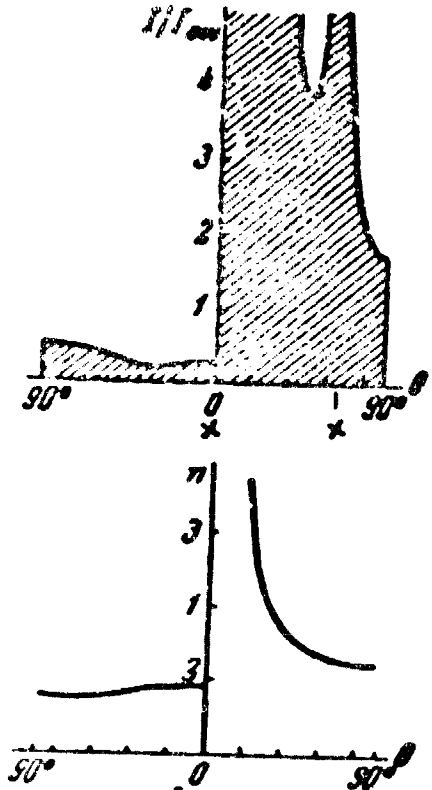
$(\omega = \omega_0 + 0)$

Fig. 15



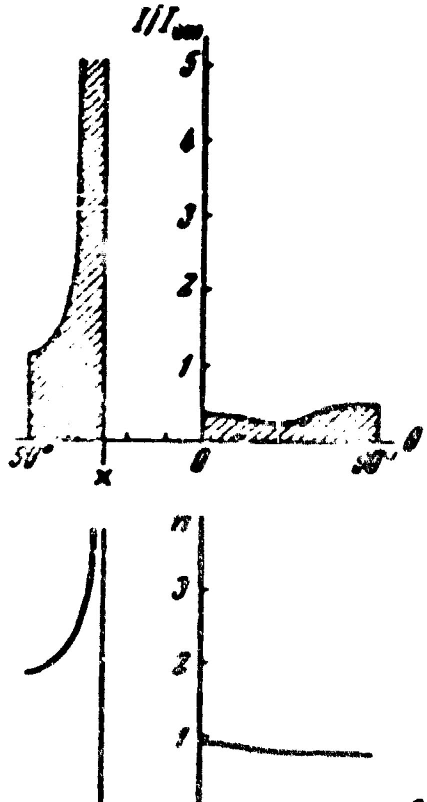
$\zeta^2 = 2/3$
$(\omega_s < \omega < \omega_H)$

Fig. 16



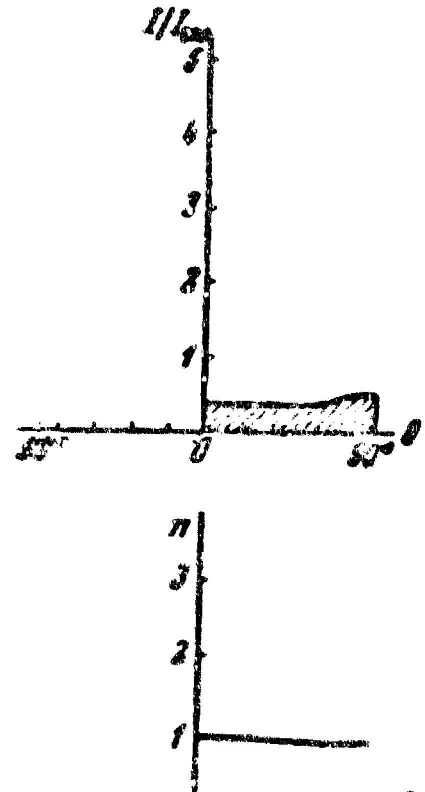
$\zeta^2 = 1 - 0$
$(\omega = \omega_H - 0)$

Fig. 17



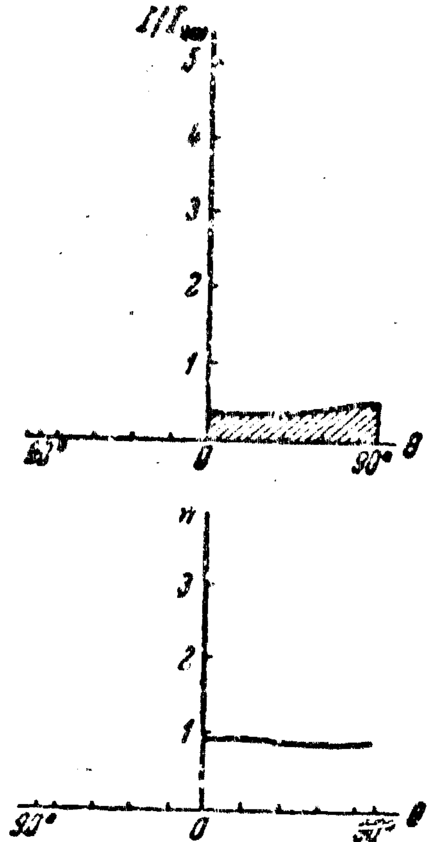
$\zeta^2 = 6/3$
$(\omega_H < \omega < \sqrt{\omega_0^2 + \omega_H^2})$

Fig. 18



$\zeta^2 = 13/9$
$(\omega = \sqrt{\omega_0^2 + \omega_H^2})$

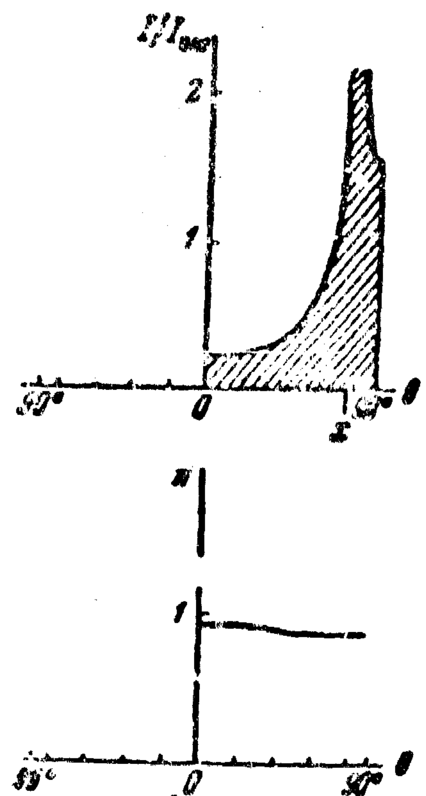
Fig. 19



$\zeta^2 = 5/3$

$$\sqrt{\omega_0^2 + \omega_H^2} < \omega < \sqrt{\omega_0^2 + \frac{\omega_H^2}{2} + \omega_H\sqrt{\omega_0^2 + \frac{\omega_H^2}{4}}}$$

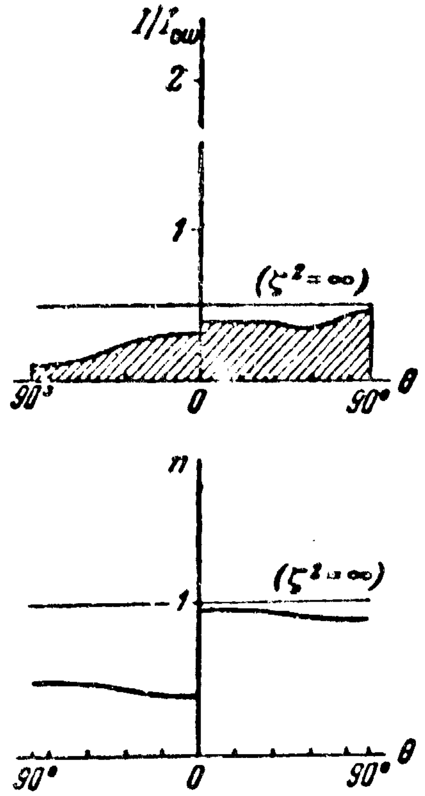
Fig. 20



$\zeta^2 = 15/9$

$$\left(\omega = \sqrt{\omega_0^2 + \frac{\omega_H^2}{2} + \omega_H\sqrt{\omega_0^2 + \frac{\omega_H^2}{4}}}\right)$$

Fig. 21



$\zeta^2=2$

$$\left(\sqrt{\omega_c^2+\frac{\omega_H^2}{2}+\omega_H\sqrt{\omega_c^2+\frac{\omega_H^2}{4}}}<\omega\right)$$

Fig. 22

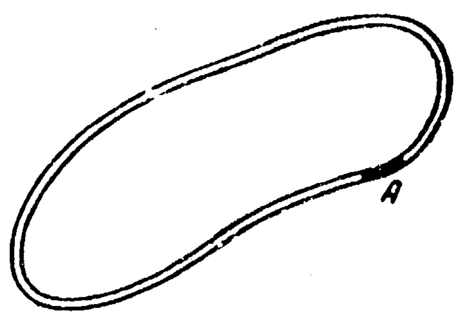


Fig. 23

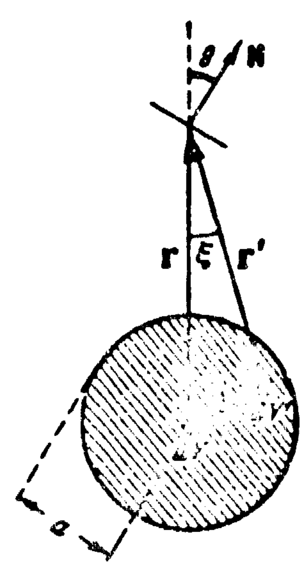


Fig. 24